STATISTICAL METHODS IN BIOLOGY, MEDICINE AND PSYCHOLOGY

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PREFACE TO FOURTH EDITION

In meeting the request of the publisher for a new edition of "Statistical Methods" it has been found necessary to review much of the extensive literature that has appeared since the first edition, published in 1899.

Among the newer developments have been the analysis of variance and extension of the theory of small samples that we owe to Dr. R. A. Fisher; and the expansion of the theory of correlation to the inclusion of multiple and partial correlations. Since some of the applications of statistics to economics include methods of general interest they have been included.

In this edition some symbols used in the earlier editions have been changed to conform with the standards of the proposed "Dictionary of Statistical Terms and Symbols" by A. K. Kurtz and H. A. Edgerton.

The present edition has received much assistance from Mr. William Drager in the computation of tables and the description of some shorter methods rendered possible by calculating machines. He has checked all computations in the most painstaking and effective manner.

For permission to reproduce tables we are indebted as follows: For Table XII copied from R. A. Fisher's "Statistical Methods for Research Workers," Oliver and Boyd, Edinburgh and London, with kind permission of author and publishers. For Table XIII, copied from G. W. Snedecor's "Calculation and Interpretation of Analysis of Variance and Covariance," Collegiate Press, Ames, Iowa, by kind permission, and with additions by Professor Snedecor. Professor Laurence H. Snyder cordially permitted us to use a large part of the twenty-ninth chapter of his excellent text book, "Principles of Heredity," D. C. Heath and Company. Also we thank Dr. A. S. Wiener for use of his table on value of Q for determining

crossing over in man. To Dr. Raymond Pearl and the W. B. Saunders Company we are indebted for permission to reproduce, in part, Appendix III of Pearl's "Medical Biometry and Statistics" as our Table XI. Dr. J. R. Miner kindly consented to our use of part of his tables of values of $1 - r^2$ and $\sqrt{1 - r^2}$. Many others have generously contributed data to make the book more generally useful.

Statisticians and other users of this book have assisted by pointing out errors in its earlier editions. A continuation of such favors with respect to the present edition is earnestly solicited.

COLD SPRING HARBOR, N. Y.
June 1, 1936

CHAS. B. DAVENPORT
MERLE P. EKAS

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STATISTICAL METHODS IN BIOLOGY, MEDICINE AND PSYCHOLOGY

CHAPTER I

On Variation and Its Measurement

Preliminary Definitions

Variation relates to individual differences in a given trait or phenomenon. It is usually due to the complex and manifold nature of the causes that reproduce the trait or phenomenon.

A variate is a single quantitative expression or magnitude determination of a character. By extension it may refer to non-quantitative expressions, such as categories of eye color. The term has been employed also to cover what is defined below as variant or class. The magnitude of any variate is designated by v (by some writers by x).

Integral variates are magnitude determinations of characters which from their nature are expressed in integers, or by counting; e.g., the number of teeth in a porpoise. These are also called discontinuous or discrete.

Graduated variates are magnitude determinations of characters which do not exist as integers and consequently in which the different variates may differ by any degree of magnitude, however small; e.g., the stature of man. These are also called continuous variates.

Variates are of two kinds: errors and variables. Errors are variates obtained by repeated measurements of the identical or constant physical phenomenon, such as the diameter of a cylinder or the velocity of light. Lack of uniformity in such measurements is largely due to errors of instrumentation, including that of the observer. Hence these variates may be called *observational* errors. Or, the single phenomenon may be changeable, like the stature of Richard Roe from morning to night. Such errors are not merely observational but also due to objective changes. Errors of this class may be called *objective* errors.

Variables are derived from the measurement of non-identical but related phenomena, like the stature of Englishmen, or the length of the spines on the body of a fly. (These may also be called collective variables.) Unlikeness of these variables is due largely to real differences between the objects measured. These variables are either of the type known as *fluctuations* and may be due primarily to the complexities of development on the one hand, or to environmentally induced changes on the other; or else they are of the type called *mutations* and are primarily genetic in origin.

A variant, among integral variates, is a single number condition; e.g., 13 ray flowers.

Variance is a measure of variation occurring in a specific group of measured traits or phenomena. It is quantitatively expressed as the mean square deviation from the mean.

A class, among graduated variates, includes variates of the same or nearly the same magnitude. The class value is designated by V. The *class range* or interval gives the limits between which the variates of any class fall. It is designated by i. Variants may also, for convenience, be grouped into classes.

Statistics deal largely with probability. A particular variate, or variant, may be regarded as the occurrence of an event, in a set of n possibilities; e.g., of a 6-fingered person in a population of 1000 persons; or of a 6-fingered person in a family of 10 characterized by much polydactylism.

Individual variation deals with diversity of a trait in a collection of individuals more or less closely related genetically.

Organ variation, or intraindividual or partial variation, deals with diversity in multiple or repeated organs in single individuals, hence generally (but not always) of genetically identical origin.

On the Collection of Data (Samples) for Statistical Analysis

Although the experimental method (in which the consequences of two set-ups practically identical except for the element whose effect is being sought are compared) is in general the most efficient method of arriving at truth, yet there are occasions when such set-ups are impracticable. In such cases statistical analysis may be of aid. Its fundamental principle is the doctrine of probability or chance. It assumes

that a sample, consisting of a large number of individuals, other objects, or events, taken without bias, or at random, from as nearly homogeneous material as possible, will practically represent the whole population in respect to the trait studied. Homogeneity, except for the single differential, is important (see discussion of the "representative method" by J. Neyman, 1934, Journal of the Royal Statistical Society, 97 : 558). If the units are divisible into two groups on the basis of a difference in effects of environment or genetics upon a given character then the consequences due to that difference may be determined, as described in Chapter IV.

To illustrate the importance of homogeneity, suppose that we wish to compare the fecundity of college women with that of non-college women. It will not do to compare white college graduates with Negro non-graduates. It will clearly be better to compare college white women with their sisters who have not gone to college. In doing so we assume that any difference in fecundity is the consequence of a single difference in set-ups; viz., "graduating from college." The value of the conclusion depends upon the validity of the assumption that "graduating from college" is the only differential between the two groups. Actually it may not be; e.g., those women who have graduated before marriage may be a lot selected on the whole for less physical attractiveness or more pressing ambition for a career, etc. Nevertheless, there are, as stated, occasions when the statistical method, with all its limitations, is the only practicable one.

The number of variates to be obtained should be large; if possible from 200 to 2000, depending on abundance and variability of the material. If, however, the numbers are small, say under 100, they can still be analyzed by special methods devised by "Student" and R. A. Fisher.

Processes Preliminary to Measuring Characters of Organisms

Some characters can best be measured directly; e.g., the stature of a race of men. Often the character can be better studied by reproducing it on paper. The two principal methods of reproducing are by photography and by camera drawings.

For photographic reproductions the organs to be measured

will be differently treated according as they are opaque or transparent. Opaque organs should be arranged if possible in large series on a suitable opaque or transparent background. The prints should be made on a rough paper so that they can be written on; blue-print paper is excellent. This method is applicable to hard parts which may be studied dry; e.g., mollusc shells, echinoderms, various large arthropods, epidermal markings of vertebrates and parts of the vertebrate skeleton. Shadow photographs may be made of the outline of opaque objects, such as birds' bills, birds' eggs and butterfly wings, by using parallel rays of light and interposing the object between the source of light and the photographic paper. More or less transparent organs, such as leaves, petals, insect wings and appendages of the smaller Crustacea, may be reproduced either directly on blue-print paper or by "solar prints," either of natural size or greatly enlarged. For solar printing the objects should be mounted in series on glass plates. They may be fixed on the plate by means of balsam or albumen and mounted between plates either dry or in Canada balsam or other permanent mounting media. Wings of flies, Orthoptera, Neuroptera, etc., may be prepared for study in this way, twenty-five to one hundred sets of wings being photographed on one sheet of paper, say 16 by 20 inches in size. Microphotographs will sometimes be found serviceable in studying small organisms or organs, such as shells of Protozoa or cytological details.

Camera drawings are a convenient although slow method of reproducing on paper greatly enlarged outlines of microscopic characters, such as the form and markings of worms and lower Crustacea, sponge spicules, bristles, scales and scutes, plant hairs, cells and other microscopic objects. In making such camera drawings a low-power objective, such as Zeiss A, will often be found very useful.

The Determination of Integral Variates; Methods of Counting

Counting offers no special difficulty with small numbers, but it becomes more difficult with an increase of numbers. To count large numbers accurately the general rule is to divide the field occupied by the numerous units into many small fields each containing only a few. It is important that

the units do not pass from one field to another during the process of counting. Among the larger examples of this type of counting are the census of population of a country, or of the stars. Among examples of small things are the blood corpuscles in a drop of blood or the organisms in a cubic centimeter of water as observed under the microscope. Eyepieces ruled into rectangles and glass slides ruled into small squares have long been used for counting microscopic objects.

The Determination of Graduated Variates; Methods of Measurement

The instruments for measuring graduated variates are of a large number of kinds. The measurements are always recorded on the metric scale since this is the universal scientific system. The fundamental units are grams, centimeters, seconds, constituting the c.g.s. system, of which the standards are preserved in Paris. There is a large number of dependent or derived units. There are units of angular measurement. There are also the absolute units of force, work, energy and power, which are derived from or dependent on gravitation, but are measured in the c.g.s. system. All measuring apparatus should be calibrated directly or indirectly with these standards. Such calibration is effected in the United States at the U. S. Bureau of Standards in Washington.

Straight lines are easily measured by means of a measuring scale of some sort. Various kinds of scales such as steel measuring tapes, graduated to millimeters (about $1.50), and steel rules (6 cm. to 15 cm.) graduated to $\frac{1}{5}$ of a millimeter can be obtained from optical companies and hardware dealers. Cloth spring tapes, 1 meter long and divided in millimeters, made in Germany, are available through the American Merchandizing Company, each good for 1000 to 2000 measurements. Steel "spring-bow" dividers with milled-head screw are useful for taking off distances which may be read off from a scale.

Instruments used in physical anthropometry are listed in Martin's "Anthropology," also in Davenport, 1927. They include (1) the anthropometer for measuring body lengths, (2) the compass caliper, (3) the sliding caliper, (4) the head height measurer, (5) the depth measurer, (6) the tape, (7) the

goniometer, (8) the balances; also many instruments for measurement of skull and small bones. Of these instruments 1, 2, 3 are made by P. Hermann, Rickenbach & Sohn, Zürich, Switzerland; Alig & Baumgärtel, Aschaffenburg, Germany; also made in America, upon inquiry of Professor T. W. Todd, Medical School of Western Reserve University, Cleveland, Ohio.

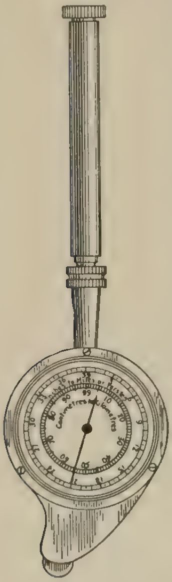

FIG. 1.—Map measurer.

Tortuous lines, e.g., the contour of the serrated margin of a leaf or the outer margin of the wing of a sphinx moth, may be measured by a map-measurer (" Entfernungsmesser," Fig. 1), supplied at artists' and engineers' supply stores at about $3.75.

Distances through solid bodies or cavities are measured by calipers made in various styles. Micrometer screw calipers (" speeded ") reading to 0.01 millimeter and sold by dealers in physical apparatus for about $5.00 are excellent for determining diameters of bones, birds' eggs, gastropod shells, etc. Leg calipers for rougher work can be obtained for 30 cents to $4.00. The micrometer "caliper-square," available for inside or outside measurements and measuring to hundredths of a millimeter, is a useful instrument.*

The area of plane surfaces, as, e.g., of a wing or leaf, is easily determined by means of a sheet of colloidin scratched in millimeter squares. By rubbing in a little carmine the scratches may be made clearer. The number of squares covered by the surface is counted (fractional squares being mentally summated), and the required area is at once obtained. If the area has been traced on paper it may be measured by the planimeter (Fig. 2). This instrument may be obtained at engineers' supply shops. It consists of two steel arms hinged together at one end; the other end of one arm is fixed

* Many of the instruments described in this section are made by the Starrett Co., Athol, Mass., and by Brown and Sharpe, Providence, R. I., tool cutters; also by Lufkin Rule Co., 106 Lafayette St., New York.

by a pin into the paper, the end of the second arm is provided with a tracer. By merely tracing the periphery of the figure whose area is to be determined the area may be read off from a drum which moves with the second arm. This method is less wearisome than the method of counting squares.

Three-dimensional Traits. These may be measured by volume or by diameters. The volume of water displaced may be used to measure volume in the case of solids. The volume of water or sand contained will measure a cavity. Irregular form is best measured from photographs or drawings. Or two or more axes may be measured and their ratio found.

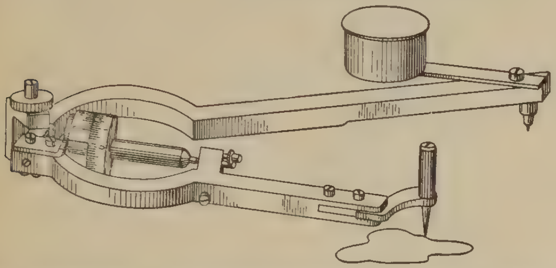

Fig. 2.—Planimeter, for determining the area by tracing the boundary of irregular plane surfaces.

The weights of bodies are determined by balances, of which there is a great variety depending on the size of the body to be measured.

Color Characters. Color may be qualitatively expressed by reference to named standard color samples. Such standard color samples are given in Ridgeway's book "Nomenclature of Color," and also in a set of samples manufactured by the Milton Bradley Company, Springfield, Mass. The best way of designating a color character is by means of the color wheel, a cheap form of which is made by the Milton Bradley Company (Fig. 3). The colors of this "top" are standard and are of known wave-length as follows (in thousandths of a micron, i.e., millionths of a millimeter):

Red......656 to 661	Green.....514 to 519
Orange....606 to 611	Blue......467 to 472
Yellow....577 to 582	Violet.....419 to 424

The red used in the color sheet is not a pure spectrum red but an "ox-blood" red, and contains a considerable proportion of black.

It is desirable to use Milton Bradley's color top as a standard.

Any color character can be matched by using the elementary colors and white and black in certain proportions. The proportions are given in percentages. In practice the fewest

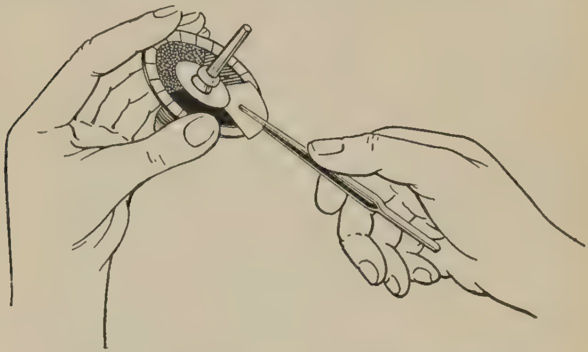

FIG. 3A.—Color top, showing how discs are adjusted.

FIG. 3B.—Discs used in the color top.

possible colors necessary to give the color character should be employed, and two or three independent determinations of each should be made at different times and the results averaged. Experience indicates that any color character is given (within 5 per cent) by only one combination of elementary colors. (See Science, July 16, 1897; Bowman, 1930; Am. J. Physic. Anthropol., 14.)

Of physiological traits a large number have been precisely measured and very delicate apparatus have been devised for the purpose. The most complete account of such apparatus

is given in E. Abderhalden's "Handbuch der biologischen Arbeitsmethoden."

Of mental traits, during the past 25 years a large number of measuring devices have been developed. Some of these are given in G. M. Whipple's "Manual of Mental and Physical Tests" (1914–1915). More recent mental tests are described in the lists of C. H. Stoelting Company of Chicago and the World Book Company of Yonkers, N. Y.—the latter largely tests of proficiency in school subjects.

Aids in Calculation

Tables. In the absence of calculating machines for multiplying and dividing, Crelle's Rechnungstafeln (Berlin: Geo. Reimer) will be found useful. The tables of Barlow ("Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals of All Integer Numbers up to 10,000") are like our Table XXIII, but more extended. A useful book of tables and formulas is that of Dunlap and Kurtz, "Handbook of Statistical Nomographs, Tables, and Formulas" (1932). Of a much more technical nature are Pearson's "Tables for Statisticians and Biometricians"; "Tables of the Incomplete Γ Function"; "Tables of the Incomplete B-Function Ratio"; and "Tables of the Complete and Incomplete Elliptic Integrals." Also the Cambridge (England) University Press publishes "Tracts for Computers." These include K. Holzinger's "Tables of the Probable Error of the Coefficient of Correlation"; and "Tables of Logarithms to 20 Decimal Places." The University of Chicago Press publishes a collection of tables by Holzinger. Very valuable are Glover's "Tables of Values of Probability Functions"; various other statistical functions, including values of the exponential growth curve, logarithms of factorial n (1 to 1000) and a table of 7-place logarithms of 5-place numbers are given (George Wahr, Michigan).

Addition of columns of numbers is best done with the aid of the "comptometer" (Fig. 4) (Felt and Tarrent Manufacturing Company, Chicago). The Burroughs machine lists and adds items. For multiplication and division of large numbers the "Monroe calculator" (Fig. 5) is excellent; machines provided with an electric motor are highly automatic. There are several other types of such computing machines.

The sorting and counting of large numbers, as in census work, is best effected by means of the Hollerith system. This

FIG. 4.—Adding machine.

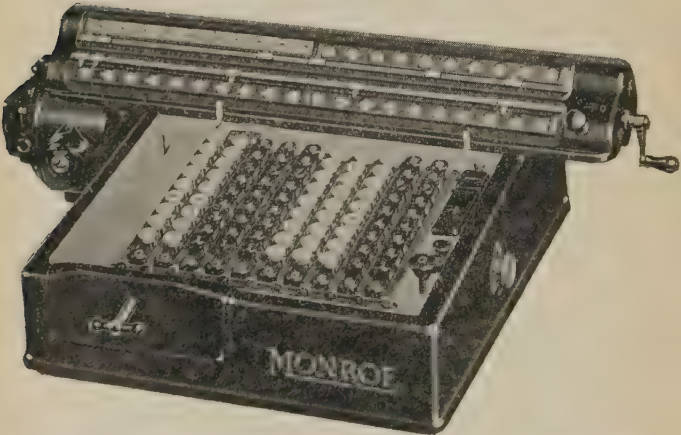

FIG. 5.—Calculator for multiplication, division, root extraction, etc.

requires the use of a special card and key punch. The sorting and counting machines may be rented from the International Business Machines Corporation (Figs. 6–8). This corporation,

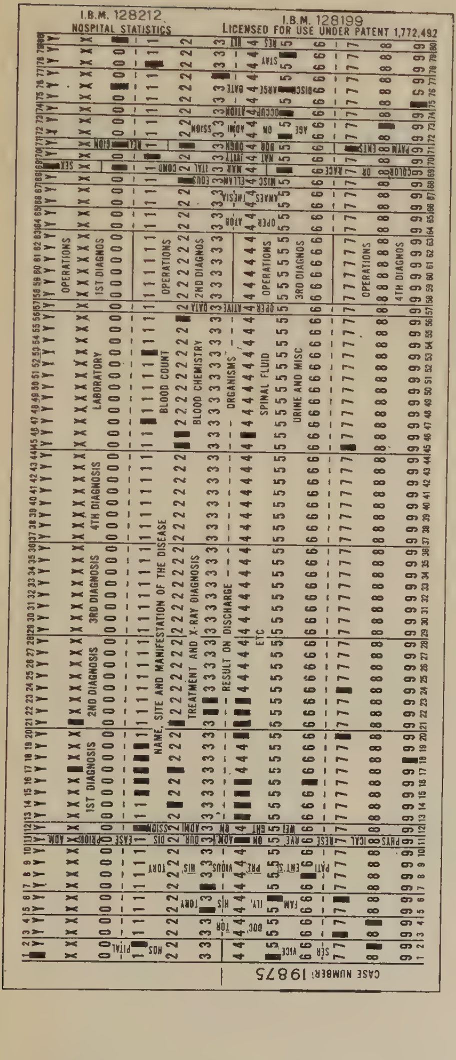

FIG. 6.—Sample of a punch card, reduced in size, used in the Hollerith machine. To use the card one has first to prepare a code in which each trait is represented by a number. If there are only 10 (or 12) traits under a head the code number is punched in 1 column. If 10 to 99 traits, each code requires 2 columns; if 100 to 999 traits each code requires 3 columns. Thus many traits (maximum 80) about an individual may be recorded on one punch card. The number of holes punched of each value for each trait is mechanically determined by the sorting and adding machines.

the Columbia University Statistical Bureau and the Recording and Statistical Corporation will furnish, at a reasonable cost, solutions to any of a wide variety of problems that are

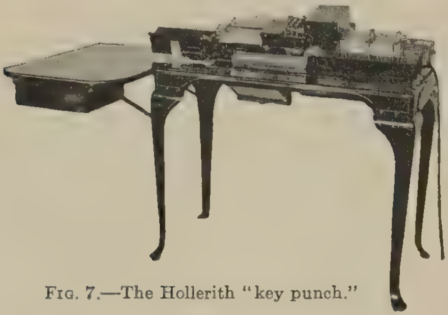

FIG. 7.—The Hollerith "key punch."

most economically handled with tabulating machine equipment.

Nomograms are graphical devices for solving equations containing two or more independent variables. Nomograms

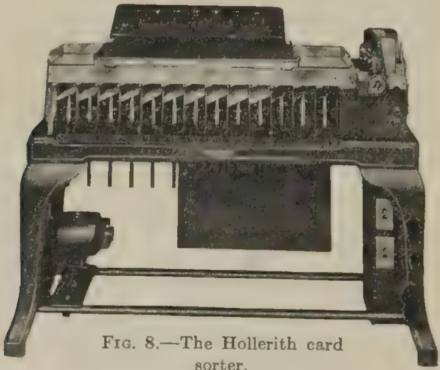

FIG. 8.—The Hollerith card sorter.

for 27 equations are given in Dunlap and Kurtz (1932). These are largely to aid in the computation of standard errors, probable errors and intelligence quotients. The slide rule is a mechanical instrument for use in solving problems similar

to some of these to which nomograms are applicable (Fig. 9). (The Macmillan Company of New York) table slide rule is a convenient, inexpensive, readily transportable device for computing power, root and circular functions.

Fig. 9.—Slide rule.

For correlation work a mimeographed checkerboard pattern on paper is a time-saving device. Numerous such correlation charts are in the market. Some of them, and other forms, are cited below.

SOME CORRELATION CHARTS

Correlation Form, with absolute checks of all operations including plotting, by R. C. Tryon, University of California. Directions and formulæ. Also form for frequency table, mean and standard deviations with checks.

E–C Correlation Chart, by M. P. Ekas and W. W. Cheyney. Engineers Pub. Co., Philadelphia. 1934. 11″ × 16½″, 21 × 18 cells 0.30″ circles. G. M. at center, summation method, both diags. used, sum formula and difference formula. Checks. Hand, machine or slide rule. Gets *r*, means, *SD*'s, regressions equations, standard errors of estimate.

Form for Correlation Coefficient and Ratios, by Karl J. Holzinger. 1928. 11″ × 17″, 24 × 24 cells 0.25″ × 0.50″. G. M. left for computer's preference. Standard formulæ. Gets *r*, means, *SD*'s, etc. Computation by logarithms.

Machine or Hand Correlation Chart, by C. H. Smeltzer. Engineers Pub. Co., Philadelphia. 1933. 8½″ × 11″, 14 × 14 cells 0.23″. G. M. at zero. Uses cross products. Hand work requires separate pieces of paper fitted to chart for writing d, d^2, products. Checks. Gets *r*, means, *SD*'s, $PE_{(SD)}$'s. Two-page directions, 13 steps, sample.

Rectangular Correlation Sheet, by Herbert A. Toops. Bureau of Publications, Teachers College, Columbia University, New York. 1922. 8½″ × 11″, 18 × 18 cells 0.25″. G. M. at zero. Diags. used. *d*'s printed. Gets *r*, means, and *SD*'s. Table of products on reverse. Separate card with window also available for products. Chart nomograph for PE_r and products of *d* steps, and their squares, by frequencies printed separately on card.

Symonds Partial and Multiple Correlation Chart, by Percival M. Symonds. Bureau of Publications, Teachers College, Columbia University, New York. 1925. Job analysis sheet for 3 variables. Entered with 3 total *r*'s, means and *SD*'s. Gets partial *r*'s, regression

coefficients, standard errors of estimate, regression equation, multiple R. Size $8\frac{1}{2}'' \times 11''$.

Precautions in Arithmetical Work

Even the most careful computers make mistakes in arithmetical work. It is absolutely necessary to take such precautions that errors may be detected. The best method is for statistical workers to compute in pairs, but independently, comparing results as the work progresses, so that time shall not be wasted by elaborate work done with erroneous values. In case of disagreement both workers should recompute, starting from that point of the work where their results check. If it is not feasible for the work to be done by two people, it should be calculated on different pages of the statistical book—proceeding through several steps on one page and then independently through the same steps on another page; checking the work as it progresses. It will be found useful as the work progresses to make rough checks by comparing the results with the original data to see that the results are probable.

Neatness in arrangement of work and in the making of figures is essential. It is best to make all calculations in a book with pages about 20 centimeters by 30 centimeters, quadruple ruled, with about three squares to the centimeter, so that each figure may occupy a distinct square. The advantage of working with a pencil is that slight errors may be erased and rectified. In case of larger errors running through several steps of the work, the erroneous calculations should not be erased but cancelled.

Upon the completion of any calculation the number of decimal places to be recorded will depend upon the probable error of each constant. It will ordinarily suffice if the probable error contain two significant figures, e.g., ± 0.17 or ± 0.0089; then the constant will be carried out to the same number of places and not farther.

ee Rulon. 1936. Sci. 84: 483-484.
a final published constant, retain no
igures beyond the position of the first
ignificant figure in one-half the
robable error (1/3 the standard error).
eep two more places in all
utations. — Better than Roessler. Sci.

CHAPTER II

On the Seriation and Plotting of Data and the Frequency Polygon

Errors and Differences

Every measurement of an invariable object (like velocity of light), every count of a large but fixed number of similar objects (like the corpuscles in a drop of blood) involves a greater or less error. When it comes to the measurement of a lot of duplicated organs on one individual (leaves on a tree, bristles on a fly) there are differences due not only to error of measurement but also to fluctuations in the developmental process. Measurements of the same dimension on related, more or less representative, individuals of a population are subject to differences that depend also, in part, on diverse genetical factors.

Simplification of Data

The result of the collection of quantitative data on one phenomenon, whether of biological or non-biological nature, is a mass of numbers which are the raw material that must be properly arranged and analyzed before one can draw important conclusions. And first of all, data must be simplified and complexes must be reduced to single numbers. Thus quotients or ratios must be computed. In the special case of a series like the percentages obtained by the color wheel, of the order: W 40 per cent, N (black) 38 per cent, Y 12 per cent, G 10 per cent, there may be selected the most variable element or that most significant for the problem in hand, e.g., N. In some cases a series of non-mensurable traits is obtained, as of eye color, and such a series, arranged in progressive order, can often be treated statistically.

Seriation and Classification

In the case of integral variates the data have to be grouped into classes that advance by integers. In the case of graduated variates, a similar grouping has to be made into classes that advance by equal intervals. The classes having been arranged in order of magnitude the number of variates occurring in each class is determined; this gives the *frequency* of the class. Each class has a *central value,* and *inner* and an *outer limiting value* and a certain *range* of values. The arrangement of frequencies by classes gives a "frequency distribution table."

The method of seriation may be illustrated by two examples: one of integral variates, and the other of graduated variates.

Example 1. The magnitudes of 22 integral variates are found to be as follows: 12, 14, 11, 13, 12, 12, 14, 13, 12, 11, 12, 12, 11, 12, 10, 11, 12, 13, 12, 13, 12, 12. In seriation they are arranged as follows:

Classes:	10,	11,	12,	13,	14.
Frequency:	1,	4,	11,	4,	2.

Example 2. In the more frequent case of graduated variates our magnitudes might be more as follows:

3.2	4.5	5.2	5.6	6.0
3.8	4.7	5.2	5.7	6.2
4.1	4.9	5.3	5.8	6.4
4.3	5.0	5.3	5.8	6.7
4.3	5.1	5.4	5.9	7.3

In this case it is clear that our magnitudes are not exact, but are merely approximations of the real (forever unknowable) value. The question arises concerning the inclusiveness of a class—the class range. One approximate rule is: Make the classes only just large enough to have no or very few vacant classes in the series. Following this rule we get:

Classes...	3.0–3.4;	3.5–3.9;	4.0–4.4;	4.5–4.9;	5.0–5.4;
	3.2	3.7	4.2	4.7	5.2
	1	2	3	4	5
Frequency	1	1	3	3	7
Classes...	5.5–5.9;	6.0–6.4;	6.5–6.9;	7.0–7.4;	
	5.7	6.2	6.7	7.2	
	6	7	8	9	
Frequency	5	3	1	1	

When the number of observations is 100 or more, the number of classes will usually run between 10 and 25. The classes are named from their middle value, or better, for ease of subsequent calculations, by a series of small integers (1 to 25).

In case the data show a tendency of the observer towards estimating to the nearest round number, like 5 or 10, each class should include one and only one of these round numbers.

As Fechner (1897) has pointed out, the frequency of the classes and all the data to be calculated from the series will vary according to the point at which we begin our seriation. Thus if, instead of beginning the series with 3.0 as in our example, we begin with 3.1 we get the series:

Classes...	3.1–3.5; 3.3	3.6–4.0; 3.8	4.1–4.5; 4.3	4.6–5.0; 4.8	5.1–5.5; 3.5
Frequency	1	1	4	3	6
Classes...	5.6–6.0; 5.8	6.1–6.5; 6.3	6.6–7.0; 6.8	7.1–7.5; 7.3	
Frequency	6	2	1	1	

which is quite a different series. Fechner suggests the rule: Choose such a position of the classes as will give a most normal distribution of frequencies. According to this rule the first distribution proposed above is to be preferred to the second.

Plotting by the Method of Rectangles

In order to give a more vivid picture of the frequency of the classes it is important to plot the frequency distribution. This is done on coördinate paper*.

The best method, especially when the number of classes is small, is to represent the frequencies by rectangles of equal base, and of altitude proportional to the frequencies. Lay off along a horizontal line (axis of X) equal contiguous spaces

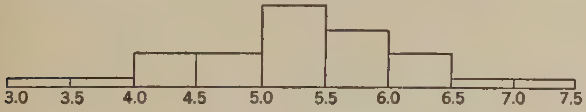

FIG. 10.—A histogram.

each of which shall represent one class; number the spaces in order from left to right with the class magnitudes in succession, and erect upon these bases rectangles whose height, proportionate to the frequency of the respective classes (Fig. 10), is given at the Y axis at the left-hand side of the graph paper.

This method of drawing the frequency distribution is known as the method of rectangles. The resulting figure is called a histogram.

* This paper may be obtained at a stationery or artists' supply store.

Plotting by the Method of Loaded Ordinates

With graduated variates, especially when the number of observations is large, the frequencies may also be represented by ordinates as follows: At equal intervals along a horizontal line draw a series of (vertical) ordinates whose successive heights shall be proportional to the frequency of the classes.

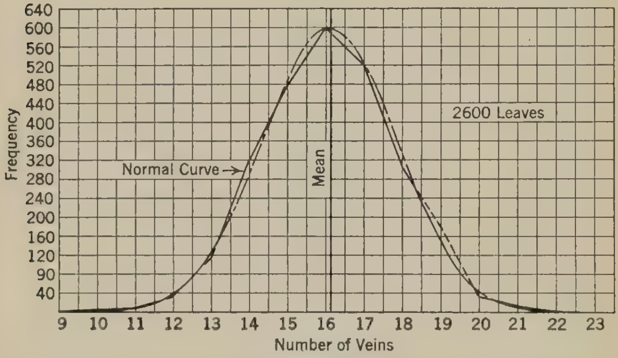

FIG. 11.—Frequency polygon, showing distribution of frequency of veins in beech leaves. After Pearson, 1902.

Join the tops of the ordinates as shown in Fig. 11. This method of drawing the frequency distribution is known as the method of loaded ordinates. The resulting figure is called the frequency polygon (*sensu strictu*). A smoothed curve of the same area may be passed through the frequency polygon; this gives the **frequency curve** (Fig. 11, broken line).

The Distribution Curve or Polygon

The distribution curve or polygon is of several forms. The most typical one is the so-called normal frequency curve, also known as the Gaussian curve. This is well illustrated by Figs. 11 and 12. It is a symmetrical curve, reaching its highest point at the mean or mode; it is concave toward the central axis as it descends on each side, but at about one-half its modal height it undergoes an inflection so that it thenceforth is convex to the central axis as the frequencies gradually decrease. Though theoretically, with an infinite number of

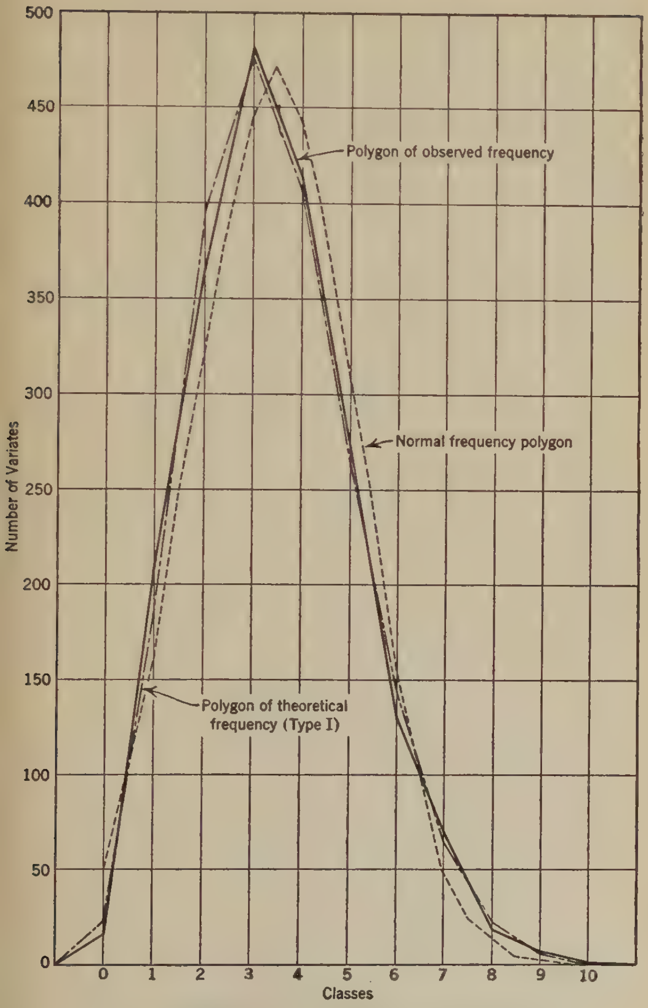

FIG. 12.—Distribution of frequency of glands in swine.

———————, polygon of observed frequency.
— – — –, polygon of theoretical frequency (Type I).
– – – – –, normal frequency polygon.

cases, the curve never quite ends on the base line but is asymptotic to the base, yet with finite numbers the Gaussian curve reaches a limit on each side of the center. Gaussian curves vary in area, height and slope of the sides. Some of the constants of this and other distribution curves are described in this chapter. Further details about the normal curve and variant curves of the same general type are given in Chapter III.

The binomial curve is that obtained by plotting as ordinates the coefficients of the successive terms of the series given by the expanded binomial $(p + q)^n$. This takes on a great variety of forms depending upon the relative size of p and q. When $p = q = \frac{1}{2}$ and n becomes indefinitely large the binomial curve is the normal (Gaussian) curve.

The Poisson Series

This series is useful in determining the distribution of occurrences (and non-occurrences) when the probability of occurrence (p) in one set is very small, but where the number of sets (n) is so great that the product np is a number of workable size.

The expected distribution of occurrences in such sets is given by the terms of the formula:

$$e^{-M}\left(1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots \frac{M^x}{x!}\right),$$

where x stands for the number of trials that are without successes.

This is very much like the binomial distribution.

The percentage of sets yielding s occurrences is

$$\frac{y}{100} = e^{-M}\frac{M^s}{s!},$$

where M is the arithmetic mean (see p. 24) and e the base of Napierian logarithms. The desired proportions for each set can be most easily found in Soper's table, published in Pearson's "Tables for Statisticians and Biometricians" (1914).

Example. Ten coins are tossed 1024 ($=n$) times, and this operation is repeated 100 times so as to give 100 samples. Problem: In what

proportion of the 100 samples (y) will 10 heads appear 2 times (s) in the 1024 tosses?

Number of times (s) in 1024 tosses that all heads appeared	Number of samples (y)
0	35
1	36
2	19
3	7
4	2
5	1
	100

The mean number of times, $M = 1.08$.

When $s = 2$,

$$\frac{y}{100} = e^{-1.08}\frac{1.08^2}{2}; \log y - \log 100 = -1.08 \log e + 2 \log 1.08 - \log 2$$

$$\log y - 2 = -0.4690 + 0.0668 - 0.3010; \quad \log y = 2 - 0.7032$$

$$\log y = 1.2968$$

$$y = 19.8 \text{ (theoretical).}$$

The Rejection of Extreme Variates

In calculating the constants of a distribution polygon, extreme variates are to be rejected only rarely and with caution. In many physical measurements Chauvenet's criterion is used to test the suspicion that a single extreme variate should be rejected. A limiting deviation ($\kappa\sigma$) is calculated. κ is the argument in Table IV corresponding to a tabular entry equal to $\frac{2n-1}{4n}$. Observations lying beyond $\kappa\sigma$ may be neglected in the computations.

Example. In 1000 minnows from one lake there are found the following frequencies of anal fin rays:

Classes:	7	8	9	10	11	12	13
Frequency:	1	2	15	279	554	144	5

$$M = 10.835 \text{ fin rays}; \quad \sigma = 0.728 \text{ fin rays.}$$

$$\kappa = \frac{1999}{4000} = 0.49975.$$

Looking in Table IV we find 3.48 corresponding to the entry 49975. Then the limiting deviation $= 3.48 \times 0.728 = 2.5334$, and the limiting class is $10.835 - 2.533 = 8.302$; hence the observation at class 7 might be excluded in calculating the

constants of the seriation; but it should not be suppressed in publishing the data.

This method depends on the improbability that any variate that exceeds a certain multiple of "σ" (see p. 29) shall properly belong to a *random sample* of the population under consideration.

Certain Constants of the Normal Frequency Polygon

After the data have been gathered and arranged it is necessary to determine the law of distribution of the variates. To get at this law we must first determine certain constants.

Measures of the Central Tendency

1. The median (*Mdn*) is the value above which and below which 50 per cent of the variates occur. To obtain it, proceed as follows: With ungrouped data arrange first in order of magnitude; if the number is odd the middle variate may be used as the median magnitude, or if the number of observations is even, use the point midway between the two central variates. For example, in the series 16, 18, 25, 27, 30, the *median* is 25; but in the series 16, 18, 20, 22, 24, 26, the median may be taken at 21.

For grouped data, the median may be found as follows:* Find the class interval in the distribution which contains the middle measure or mid-score and interpolate to find the probable score of this median measure, assuming that the measures in the class interval are evenly distributed.
The formula is:

$$Mdn = ll + i\left(\frac{\frac{N}{2} - f_{(cum)}{}^{\mathrm{UP}}}{f_{mdn}}\right)$$

---

* Midpoint is dependent on choice of class interval limits. For example, 42.5 is midpoint of an interval assumed to begin at 40.00 and end at 44.99+. The limits of an interval should be chosen by the experimenter to fit his data in accordance with the technique of taking the original measurements. For example, if a weight measurement is precisely 40, such weight would be placed in a class having the limits 40.00 and 44.99+, inclusive. If, however, the experimenter takes each weight to the *nearest* kilogram, so that 39.6, for example, would be recorded as 40, the limits of the interval would become 39.50 to 44.49+, with the midpoint at 42.00.

or
$$Mdn = ul - i\left(\frac{\frac{N}{2} - f_{(cum)}^{\text{DOWN}}}{f_{mdn}}\right)$$

where Mdn = median.
ll = lower limit of middle class interval.
ul = upper limit of middle class interval.
i = width of class intervals.
$f_{(cum)}^{\text{UP}}$ = cumulative frequency counted UP to lower limit of middle class interval.
$f_{(cum)}^{\text{DOWN}}$ = cumulative frequency counted DOWN to upper limit of middle class interval.
f_{mdn} = frequency of interval containing median.

Example.

Weight in kilos	f	$f_{(cum)}^{\text{UP}}$	$f_{(cum)}^{\text{DOWN}}$
65–69	1		1
60–64	2		3
55–59	4		7
50–54	7		14
45–49	15		Median interval
40–44	7	27	
35–39	11	20	
30–34	5	9	
25–29	4	4	
	$N = 56$		

In the distribution above, the middle measure is $\frac{N}{2}$ or $\frac{56}{2} = 28$. Counting up from the bottom of the distribution, the median falls in the interval 45–49. Interpolating, the median is $\frac{1}{15}$ of the way between 45–50. Then $45 + (\frac{1}{15} \times 5) = 45.33$. This can be checked by counting and interpolating in the other direction, i.e., down, whence

$$50 - (\tfrac{14}{15} \times 5) = 45.33.$$

The median has the advantage that it can be computed with accuracy when the more extreme variates are known only roughly. It can often be determined by inspection of the series. It is especially useful with erratically spaced and non-quantitative variates. Unlike the mean it is little affected by extreme variates due to special cases like the stature of a dwarf or giant included in a series of stature measures. Irving Fisher (1922, p. 260) finds that it is rather better than any other simple form of average in computing index numbers.

2. The Arithmetic Mean (M) is the sum of the separate variates divided by the number of variates. For *ungrouped data* the mean is computed as follows:

$$M = \frac{\Sigma v}{N}.$$

M, mean of the distribution.
Σ, the sum of.
v, individual variates.
N, number of variates.

Example.

Individual	v (Weight in pounds)
1.	143
2.	132
3.	117
4.	128
5.	140
$N = 5$	$\Sigma v = 660$

$M = \frac{660}{5} = 132$ lb.

This method is impracticable with long series unless a calculating machine is available.

For *grouped data:*

$$M = \frac{\Sigma fV}{N}$$

where M = mean of the distribution.
Σ = the sum of.
f = the frequency of each class.
V = midpoint of each class.
N = number of variates.

Example.

Weight in kilos	V	f	fV
65–69	67.5	1	67.5
60–64	62.5	1	62.5
55–59	57.5	5	287.5
50–54	52.5	4	210.0
45–49	47.5	8	380.0
40–44	42.5	13	552.5
35–39	37.5	11	412.5
30–34	32.5	7	227.5
25–29	27.5	5	137.5
		$N = 55$	$\Sigma fV = 2337.5$

$$M = \frac{2337.5}{55}$$

$$M = 42.5$$

This formula involves tedious labor when N is large unless a calculating machine is available.

To simplify the calculation with *grouped data* proceed as follows:

$$M = V_0 + i\,\frac{\Sigma f(V - V_0)}{N}.$$

V_0 = assumed mean.
f = frequency of each class.
$V - V_0$ = deviation of class step from assumed mean.
i = class range.
N = number of cases or variates.

Example.

Weight in kilos	f	$V - V_0$	$f(V - V^0)$
65–69	1	4	4
60–64	1	3	3
55–59	5	2	10
50–54	4	1	4
45–49	8	0	0
40–44	13	−1	−13
35–39	11	−2	−22
30–34	7	−3	−21
25–29	5	−4	−20
	$N = 55$		$\Sigma f(V - V_0) = -55$

$$M = 47.5 + 5 \times \frac{-55}{55} = 42.5.$$

A formula based on *cumulative frequency sums;*

$$M = V_0 - i\left(\frac{\Sigma f_{(cum)}{}^{\text{UP}}}{N}\right)$$

where V_0 = assumed mean taken at midpoint of highest interval plus one class.
$f_{(cum)}{}^{\text{UP}}$ = the cumulative frequency from the bottom up.

Example.

Weight in kilos	f	$f_{(cum)}{}^{\text{UP}}$
65–69	1	55
60–64	1	54
55–59	5	53
50–54	4	48
45–49	8	44
40–44	13	36
35–39	11	23
30–34	7	12
25–29	5	5
	$N = 55$	$\Sigma f_{(cum)}{}^{\text{UP}} = 330$

$$M = 72.5 - 5 \times \tfrac{330}{55};\quad M = 42.5.$$

This method is especially adaptable for adding or calculating machines. An efficient check is provided by summing in the opposite direction and adding the correction to the midpoint of the lowest interval minus one step. The formula then becomes

$$M = V_0' + i\frac{f_{(cum)}{}^{\text{DOWN}}}{N}$$

where V_0' = one class interval below midpoint of lowest class.
$f_{(cum)}{}^{\text{DOWN}}$ = cumulative frequency counted down from top.
i = width of class interval.

Example.

Weight in kilos	f	$f_{(cum)}{}^{\text{DOWN}}$
65–69	1	1
60–64	1	2
55–59	5	7
50–54	4	11
45–49	8	19
40–44	13	32
35–39	11	43
30–34	7	50
25–29	5	55

$N = 55 \quad f_{(cum)}{}^{\text{DOWN}} = 220$

$M = 22.5 + 5 \times \frac{220}{55}$.

$M = 42.5$.

3. **The geometric mean** of a series of variates is the Nth root of the product of all terms in the series.

$$G = \sqrt[N]{v_1 \cdot v_2 \cdot v_3 \cdot v_4 \ldots}$$

By logarithms,

$$\log G = \frac{\Sigma(\log v)}{N}.$$

If the variates are grouped into classes whose midpoints are V_1, V_2 V_3, etc., and whose corresponding frequencies are f_1, f_2, f_3, etc.

$$G = \sqrt[N]{V_1^{f_1} \cdot V_2^{f_2} \cdot V_3^{f_3} \ldots}$$

where N equals the total frequency. This computation is carried out with the aid of logarithms as follows:

$$\log G = \frac{f_1 \log V_1 + f_2 \log V_2 + f_3 \log V_3 + \ldots}{N}$$

where G = geometric mean.
N = number of variates.
v_1, v_2, etc. = the individual variates.
V_1, V_2, etc. = the mid-class values.
f_1, f_2, etc. = the frequencies of the corresponding classes.

The geometric mean is always less than the arithmetic mean and greater than the harmonic mean (see below). It is especially useful in averaging rates of change, ratios and series of values increasing in geometric fashion.

Example. To determine the geometric average weight of the intra-uterine mouse from the observed average weights of embryos at various ages.

Embryo age in days	Average weight	Variates
3	0.0086	v_1
4	0.0329	v_2
5	0.0762	v_3
6	0.1298	v_4
7	0.2288	v_5
8	0.3651	v_6
9	0.5926	v_7
10	0.8467	v_8
11	1.190	v_9

Computation with logs:

$$
\begin{aligned}
\log 0.0086 &= 7.9344985 - 10\\
\log 0.0329 &= 8.5171959 - 10\\
\log 0.0762 &= 8.8819550 - 10\\
\log 0.1298 &= 9.1132747 - 10\\
\log 0.2288 &= 9.3594560 - 10\\
\log 0.3651 &= 9.5624118 - 10\\
\log 0.5926 &= 9.7727616 - 10\\
\log 0.8467 &= 9.9277296 - 10\\
\log 1.190 &= 0.0755470\\
\hline
&\quad 73.1448301 - 80\\
&\quad 10.0000000 - 10\\
\hline
&\quad 83.1448301 - 90
\end{aligned}
$$

$$\log G = \frac{83.1448301 - 90}{9} = 9.2383144 - 10.$$

$$G = 0.1731.$$

4. **The harmonic mean** is the reciprocal of the arithmetic mean of the reciprocals of the variates.

$$M_H = \frac{1}{\frac{1}{N}\sum \frac{1}{v}}$$

where M_H = harmonic mean

$\frac{1}{N}$ = reciprocal of number of variates.

$\sum\left(\frac{1}{v}\right)$ = sum of reciprocals of variates.

In the series 18, 20, 21, 21, 25, 32, 35, M_H is given by

$$M_H = \frac{1}{\frac{1}{7}\left(\frac{1}{18}+\frac{1}{20}+\frac{1}{21}+\frac{1}{21}+\frac{1}{25}+\frac{1}{32}+\frac{1}{35}\right)}$$

$$M_H = \frac{1}{\left\{\begin{matrix}0.14286\ (0.0556 + 0.0500 + 0.0476 \\ +0.0476+0.040+0.0313 + 0.0286)\end{matrix}\right\}}.$$

$$M_H = \frac{1}{0.042958} = 23.28.$$

The harmonic mean is always less than either the geometric or the arithmetic mean. It is especially adapted to the averaging of such items as time rates.

5. The mode (M) is the class with the greatest frequency. It is necessary to distinguish sharply between the empirical and the theoretical mode. The *empirical mode* is that mode which is found on inspection of the seriated data. In the example on page 24 the empirical mode is 42.5 kilos. The *theoretical mode* is the mode of the theoretical curve most closely agreeing with the observed distribution. Pearson (1902, p. 261) gives this rule for roughly determining the theoretical mode. The mode lies on the opposite side of the median from the mean; and the abscissal distance from the median to the mode is double the distance from the median to the mean; or, mode = mean − 3 × (mean − median). More precise directions for finding the mode in the different types of frequency polygons are given in the discussion of the types.

6. The simple aggregative index is used in place of a mean in securing an index number in economics. It is obtained by taking the sum of the actual prices of certain commodities for a given year and dividing this by the sum of the prices of the same commodities for a base year. The formula for the aggregative index number is $\frac{\Sigma p_1}{\Sigma p_0}$, Σp_1 being the sum of prices for a given year and Σp_0 for the base year. The best index number as proposed by Irving Fisher (1922, p. 482) is

Σ — large sigma

one in which quantity of production (q) is brought into the equation. Thus the weighted arithmetic mean is

$$\frac{\Sigma p_0 \cdot q_0 \left(\frac{p_1}{p_0}\right)}{\Sigma p_0 \cdot q_0} \tag{1}$$

A weighted harmonic mean is $\dfrac{\Sigma p_1 q_1}{\Sigma p_1 \cdot q_1 \left(\frac{p_0}{p_1}\right)}$. (2)

The "ideal" index is $\sqrt{(1) \times (2)}$.

Measures of Variability

1. The **range** is the total spread of the distribution from lowest to highest measure; e.g., the distribution 5, 7, 8, 10, 13 has a *range* 13 − 5, or 8. The *range* is simple to compute, but tells nothing about the shape of the distribution or the concentration about the central tendency. small sigma

2. The **standard deviation,** σ, is the square root of the mean of the squares of the deviations from the mean, or the square root of the variance.

For *ungrouped data:*

$$\sigma = \sqrt{\frac{\Sigma(v - M)^2}{N}}$$

Better N − 1

where σ = standard deviation of the distribution.
$\Sigma(v - M)^2$ = sum of the squared deviations from the mean.
N = number of cases.

See p. 31

Example.	Individual	Measure	$v - M$	$(v - M)^2$
	1.	132	+11	121
	2.	116	− 5	25
	3.	124	+ 3	9
	4.	102	−19	361
	5.	140	+19	361
	6.	115	− 6	36
	7.	118	− 3	9
	$N = 7$	847	$\Sigma(v - M)^2 =$	922

Mean = $\frac{847}{7}$ = 121. $\sigma = \sqrt{\frac{922}{7}} = 11.48$.

For *grouped data:*

$$\sigma = \sqrt{\frac{\Sigma f(V - M)^2}{N}}$$

where $f(V - M)^2$ = for each class, the squared deviation multiplied by the frequency.

Example.

	f	$V - M$	$(V - M)^2$	$f(V - M)^2$
100–109	3	26.44	699.0736	2097.2208
90–99	6	16.44	270.2736	1621.6416
80–89	11	6.44	41.4736	456.2096
70–79	14	− 3.56	12.6736	177.4304
60–69	7	−13.56	183.8736	1287.1152
50–59	3	−23.56	555.0736	1665.2208
40–49	1	−33.56	1126.2736	1126.2736

$N = 45$ $\Sigma f(V - M)^2 = 8431.1120$

Mean = 78.56. $\sigma = \sqrt{\dfrac{8431.1120}{45}} = 13.69.$

For *computation with assumed mean:*

$$\sigma = i\sqrt{\frac{\Sigma f(V - V_0)^2}{N} - \left[\frac{\Sigma f(V - V_0)}{N}\right]^2}$$

where i = class interval.

$f(V - V_0)$ = for each class, the deviation multiplied by the frequency.

$f(V - V_0)^2$ = for each class, the squared deviation multiplied by the frequency.

Example.

V	f	$(V - V_0)$	$f(V - V_0)$	$f(V - V_0)^2$
100–109	3	3	9	27
90–99	6	2	12	24
80–89	11	1	11	11
70–79	14	0		
60–69	7	−1	− 7	7
50–59	3	−2	− 6	12
40–49	1	−3	− 3	9

$N = 45$ $\Sigma f(V - V_0) = +16$; $\Sigma f(V - V_0)^2 = 90$

$$\sigma = 10\sqrt{\frac{90}{45} - \left(\frac{16}{45}\right)^2}.$$

$$\sigma = 13.69.$$

The *cumulative sums method for* σ:

$$\sigma = i\sqrt{\frac{2\,\Sigma f''}{N} - \frac{\Sigma f'}{N} - \left(\frac{\Sigma f'}{N}\right)^2}$$

where f' = the cumulative frequency from the bottom up to and including that of the given class.

f'' = the cumulative frequencies of the f''s up to and including that of each class.

Example.

V	f	f'	f''
100–109	3	45	164
90–99	6	42	119
80–89	11	36	77
70–79	14	25	41
60–69	7	11	16
50–59	3	4	5
40–49	1	1	1
	N = 45	164	423

$$\sigma = 10\sqrt{\frac{2(423)}{45} - \frac{164}{45} - \left(\frac{164}{45}\right)^2}.$$

$$\sigma = 13.69.$$

3. The Estimation of the Squared Standard Deviation, or "Variance," from Small Samples (s). As shown in paragraph 2, σ is computed from the sum of the squares of deviations of each variate from the mean divided by the number of variates. This assumes that the number of the deviations is the same as the degrees of independence, or "freedom," of the cases, taken collectively. If the number of cases is large the result obtained on this assumption is approximately correct. But if the number of cases is small the fact must be taken into account that the number of free deviations is reduced by 1, owing to the fact that the algebraic sum of all deviations from the mean is 0. Consequently, in getting the average of the squared deviation it is necessary to divide by $N - 1$. Thus in the illustration on page 29, of 7 measurements (ungrouped), variance is, strictly, $s^2 = \frac{922}{7-1} = \frac{922}{6} = 153.67$, hence $s = 12.40$, instead of 11.48.

4. The quartile deviation, semi-interquartile range, or Q, is one-half the difference between the first and third quartiles.

$$Q = \frac{Q_3 - Q_1}{2}$$

where Q = quartile deviation.

Q_3 = third quartile, or 75 centile.

Q_1 = first quartile, or 25 centile.

Example.

Score	f	$f(cum)$
25–29	5	70
20–24	13	65
15–19	25	52
10–14	15	27
5–9	10	12
0–4	2	2
	70	

f = frequency at each step.
$f(cum)$ = cumulative frequency.

Quartiles are computed in similar manner as the median which is the second quartile, i.e., Q_2.

Q_3:

$$0.75(70) = 52.50$$
$$52.50 - 52 = 0.50$$
$$20 + \left(\frac{0.50}{13} \times 5\right) = 20.19.$$

Q_1:

$$0.25(70) = 17.50$$
$$17.50 - 12 = 5.50$$
$$10 + \left(\frac{5.50}{15} \times 5\right) = 11.83.$$

$$Q = \frac{20.19 - 11.83}{2} = 4.18.$$

The value $Q = 4.18$ is interpreted to mean that, in the above distribution, the middle half of the scores lie within 4.18 score points of the median, i.e., $\pm$ 4.18.

5. The average deviation is the average (regardless of sign) of the individual deviations from a central point of tendency of the distribution.

For *ungrouped data:*

$$AD = \frac{\Sigma \mid x \mid}{N}$$

where AD = average deviation.
$\Sigma \mid x \mid$ = sum of individual deviations from mean without regard to sign.
N = number of variates.

Example.

Individual	Variate	x	$v - M$
1.	132	+11	(132 − 121)
2.	116	− 5	(116 − 121)
3.	124	+ 3	etc.
4.	102	−19	
5.	140	+19	
6.	115	− 6	
7.	118	− 3	
$N = 7$	847	$\Sigma \mid x \mid = 66$	

Mean $= \frac{847}{7} = 121.$ $AD = \frac{66}{7} = 9.43.$

For *grouped data:*

$$AD = \frac{\Sigma \mid fx \mid}{N}; \text{ where } x = V - M.$$

Example.

	f	$(V - M)$	$f(V - M)$	
100–109*	3	26.44	79.32	
90–99	6	16.44	98.64	
80–89	11	6.44	70.84	
70–79	14	− 3.56	−49.84	(Mean = 78.56)
60–69	7	−13.56	−94.92	
50–59	3	−23.56	−70.68	
40–49	1	−33.56	−33.56	
N =	45	$\Sigma \mid f(V - M) \mid$ =	497.80	

$$AD = \frac{497.80}{45} = 11.06.$$

The average deviation of a normal distribution marks the limits of the middle 57.5 per cent of the measures. Applied to the accompanying problem (although not a normal distribution), these limits are 78.56 ± 11.06 = 89.62 to 67.50.

Coefficient of Variation. The standard deviation, like the other indices of variation, is a concrete number, being expressed in the same units as the magnitudes of the classes. The standard deviation of one lot of variates is consequently not comparable with the standard deviation of variates measured in other units. The index of variation may be reduced to an abstract number, independent of any particular unit, by dividing the index of variation, σ, of any set of variates by their mean value. The quotient multiplied by 100 is called the coefficient of variation. In a formula,

$$C = \frac{\sigma}{M} \times 100 \text{ (Pearson, 1896; Brewster, 1897).}$$

Pearl (1927) has suggested a neat graphic representation of variability relative to the mean, in which the abscissæ are the mid-class values expressed in percentage of the mean value. The ordinates are the absolute frequencies of each class, per 1 per cent of mean. For example, see Table 1: Variability

TABLE 1.—ABSOLUTE AND RELATIVE FREQUENCY DISTRIBUTIONS FOR VARIATION IN (*a*) STATURE, (*b*) BODY WEIGHT AND (*c*) PULSE RATE, OF A SAMPLE OF MAYA INDIANS, BASED ON DATA OF M. STEGGERDA (1932)

Stature

Class, in cm.	2	3	4	5	6
145–149.99	14	182	95.09	4.34	56.36
150–154.99	24	312	98.32	7.45	96.75
155–159.99	26	337	101.54	8.07	104.81
160–164.99	10	130	104.76	3.10	40.26
165–169.99	3	39	107.99	0.93	12.08
	77	1000			

$M = 155.11$; $\sigma = 5.25$; C. V. = 3.38.

Body Weight

Class, in kilograms	2	3	4	5	6
41–42.99	1	15	78.45	0.27	3.91
43–44.99	3	44	82.18	0.80	11.59
45–46.99	2	29	85.92	0.54	7.83
47–48.99	8	116	89.65	2.14	31.01
49–50.99	10	144	93.39	2.68	38.84
51–52.99	17	246	97.12	4.55	65.94
53–54.99	9	130	100.86	2.41	34.93
55–56.99	6	87	104.59	1.61	23.33
57–58.99	5	73	108.33	1.34	19.42
59–60.99	2	29	112.07	0.54	7.83
61.62.99	3	44	115.80	0.80	11.59
63–64.99	0	....	119.54	0	0
65–66.99	0	....	123.27	0	0
67–68.99	2	29	127.01	0.54	7.83
69–70.99	0	....	130.74	0	0
71–72.99	1	15	134.48	0.27	3.91
	69	1001			

$M = 53.54$; $\sigma = 5.54$; C. V. = 10.35

Pulse Rate

Class, in number per minute	2	3	4	5	6
29.50–34.49	1	11	61.15	0.10	1.09
34.50–39.49	2	22	70.71	0.21	2.28
39.50–44.49	4	44	80.26	0.42	4.57
44.50–49.49	23	250	89.81	2.41	26.20
49.50–54.49	35	380	99.37	3.66	39.78
54.50–59.49	17	185	108.92	1.78	19.35
59.50–64.49	5	54	118.48	0.52	5.65
64.50–69.49	2	22	128.03	0.21	2.28
69.50–74.49	1	11	137.59	0.10	1.09
74.50–79.49	2	22	147.14	0.21	2.28
	92	1001			

$M = 52.33$; $\sigma = 7.25$; C. V. = 14.12.

2 Frequency.
3 Frequency, per mille.
4 Per cent that class midpoint is of mean.
5 Absolute frequency per 1 per cent of mean.
6 Per mille frequency per 1 per cent of mean.

of Maya Indians in stature, weight and pulse rate (data from Steggerda, 1932, pp. 11–19, 91). See Fig. 13.

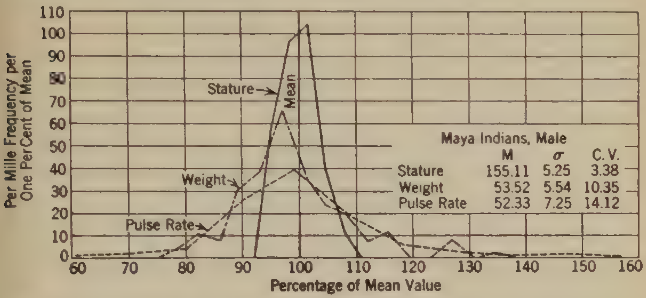

FIG. 13.—Graphic representation of variability of stature, weight and pulse rate in Maya Indians. See text

Sampling and Its Errors

The traits of any population of individuals, whether people or animals or plants or values, are variable. To find the variability of any trait in the entire population is generally impracticable because of its extent. One must take a fraction of the population to stand for the whole, and so is raised the necessity of studying a sample and letting its constants stand for those of the whole population. The justification for so doing rests on the laws of probability.

Probability. If from a bag of perfectly shuffled balls of which 1000 are white, 500 black, 300 red, one draws 90 (replacing each as drawn and shuffling), then it is easy to compute what the probability will be of each color. Or, conversely, if we draw 50 white, 25 black and 15 red, by the laws of probability it may be inferred, as the best bet, that in a bag of 1800 balls 1000 are white, 500 black and 300 red. However, in general, the larger the sample and the less variable the population the more reliance can be placed on such inferences.

The probability that an event will happen is usually designated by p; that it will fail by $(1 - p) = q$, certainty being a probability of $p + q = 1$. Also, the probability of occurrence together of independent events is equal to the product of the probabilities of the independent events. In a number of inde-

pendent events which, taken separately, have a certain probability of occurrence, the probability of one or other occurring is the sum of the separate probabilities. Again, if p is the probability that an event will occur in one trial, p^n is the probability that it will happen n times in succession.

Upon the principles of probability is rendered our judgment of the sufficiency or reliability of any set of data to give significant or representative constants.

If instead of one large sample a number of smaller samples be obtained and their means found, these means will have a "sampling distribution" which follows the normal law. The mean of this sampling distribution will be close to the mean of the entire population. Its variability and the deviation of the mean from the true mean may be measured as indicated below.

The **probable error,** PE, of the determination of any value gives the measure of unreliability of the determination. The probable error is a pair of values lying one above and the other below the constant determined. There is an even chance that the true value lies between these limits, provided the distribution for which the constant was obtained is normal or nearly normal. The chances that the true value lies within

$\pm 2PE$ are 4.6 : 1.	$\pm 5PE$ are 1,341 : 1.
$\pm 3PE$ are 22 : 1.	$\pm 6PE$ are 19,300 : 1.
$\pm 4PE$ are 142 : 1.	$\pm 7PE$ are 427,000 : 1.

$\pm 8PE$ are 14,700,000 : 1.
$\pm 9PE$ are 730,000,000 : 1.

The probable error should be found to two significant figures for most data. The constant for which it has been determined should be carried to the same number of places (see p. 14). The probable error of a constant is usually written after the value of the constant to which it is linked by the symbol $\pm$.

The **standard error** is $1/0.6745$ (nearly 1.5) times the probable error. Though the probable error is so firmly established that it may be said to be the prevailing mode of representing the unreliability of a constant, the standard error will probably in time prevail, since it avoids the necessity of applying the factor 0.6745. When the standard error follows a constant it would be well to join the two numbers by the symbol $\mp$, to distinguish it from the probable error.

The Probable Errors and Standard Errors of Leading Constants. The probable error of the mean:

$$PE_M = \pm \frac{0.6745\sigma}{\sqrt{N}}.$$

The standard error of the mean:

$$SE_M = \mp \frac{\sigma}{\sqrt{N}}.$$

The probable error of the median:

$$PE_{Mdn} = \pm \frac{0.8454\sigma}{\sqrt{N}}.$$

The probable error of the standard deviation:

$$PE_\sigma = \pm \frac{0.6745\sigma}{\sqrt{2N}}.$$

The standard error of the standard deviation:

$$SE_\sigma = \mp \frac{\sigma}{\sqrt{2N}}.$$

The probable error of the coefficient of variation:

$$PE_C = \pm \frac{0.6745C}{\sqrt{2N}}\left[1 + 2\left(\frac{C}{100}\right)^2\right]^{1/2}.$$

When C is small, say less than 10 p.c., the factor within brackets may be omitted, especially since only two significant figures of the probable error need be recorded.

To Measure the Significance of a Difference between Means. Means obtained from two independent samples of the same universe will rarely be precisely alike. A small difference in the means may not indicate a real difference.

To determine whether or not a difference between means is probably significant, find the square root of the sum of the squares of the two standard errors of the means. This root is the standard error of the difference between the two means. If the difference between the means exceeds twice the root, the difference is probably significant.

Significant differences are those in excess of

$$3\sqrt{PE_{Mx}^2 + PE_{My}^2} \quad \text{or of} \quad 2\sqrt{SE_{Mx}^2 + SE_{My}^2}.$$

Of course, the more the difference between means exceeds the limit of 2 × the standard error of the means the more certainly significant the difference becomes.

In the case of small samples, obtain in each sample the sum of the squares of deviations from its mean and add together such sums; divide by the sums of the numbers of the degrees of freedom (in this case: $[N_1 - 1] + [N_2 - 1]$). This will give the square of the approximate σ, or call it s^2. Then if the quotient of

$$\frac{M_1 - M_2}{s\sqrt{\frac{1}{N_1} + \frac{1}{N_2}}} = t$$

is greater than the value in fine print (Table XIII) the difference is significant (Tippett, 1931, p. 81).

The probable error of a difference between two correlated means:

$$PE(M_x - M_y) = \sqrt{PE_{Mx}^2 + PE_{My}^2 - 2r_{M_xM_y}PE_{M_x}PE_{M_y}}.$$

To Measure the Differences between Variabilities. The general rule is to take the square root of the sum of the squared standard errors in each series. If the difference between the means exceeds twice the root this difference is probably significant.

To Measure the Differences between Variabilities of Two Small Samples. R. A. Fisher (1932, p. 206) suggests using the formula $z = \log_e \frac{\sigma_1}{\sigma_2}$, where σ_1 and σ_2 are variabilities calculated with regard to degrees of freedom (p. 31), and z is the required relation between the variabilities. This equation may be put in form $z = \frac{1}{2}(\log_e \sigma_1^2 - \log_e \sigma_2^2)$, which avoids the operation of division.

Snedecor (1934, p. 15) proposes the formula

$$F = \frac{\text{larger mean square}}{\text{smaller mean square}} = \frac{\sigma_1^2}{\sigma_2^2}$$

and affords a table (reproduced in our Table XIII) of values of F based upon degrees of freedom of the two samples by

which one can infer the value of F that would be probably significant of a real difference; and that would *almost certainly* be significant of such a real difference, beyond the chance of errors of sampling (see also p. 70).

Quick Methods of Roughly Determining Average and Variability. 1. Arrange the specimens (e.g., of a series of beech leaves) in a series according to the magnitude of the character, simply judging the order by the eye. Then pick out those two that will divide the series into thirds and measure them. Their average will be the average of the whole series. Then,

$$\frac{\text{mean} - \text{the smaller of the two measures}}{0.43} = \sigma$$

(0.43 is the value of $\pm x$, at which the area of the curve included between these limits of x equals one-third of the whole).

Or, 2. Select roughly two specimens that seem to be about one-third of the distance from the two extremes and group all others as larger than the larger one, smaller than the smaller one or between the two. Measure the two specimens. Count the number in each group and determine σ by aid of Table IV (p. 166) as follows: Taking as origin the middle of the whole series, call the number of leaves from the middle to the smaller n_2', and the number from the middle to the larger n_2''; also, the x distance to the lower division point h_1 and to the upper division point h_2. Then $(h_1 + h_2)$ = the range covered by the middle division or the difference between the upper and lower value. As we know the areas of the curve between the origin and h_1 on the one hand and h_2 on the other (percentage of individuals between the middle and h_1 and h_2), we can find $\frac{h_1}{\sigma}$ and $\frac{h_2}{\sigma}$ from Table IV, since they are the values $\frac{x}{\sigma}$ corresponding to the percentage areas determined. But $\frac{h_1}{\sigma} + \frac{h_2}{\sigma} = \frac{(h_1 + h_2)}{\sigma}$; thus σ is determined. Knowing σ we can get h_1 or h_2, and hence the mean. Or the value of the character of the middle specimen may be taken as the mean value.

Example. Seventy-six beech-leaves which had fallen from one tree were picked up. They were sorted out as in the second method. It was found that 22 were smaller than the smaller type leaf, which was

1.78 inches in length; and 23 were larger than the larger type leaf (2.22 inches in length). The 38th leaf is the middle of the series, and so the smaller type leaf was distant 16 leaves from the middle, and the larger 15.

$$\frac{n_2'}{n} = \frac{16}{76} = 0.2105; \qquad \frac{n_2''}{n} = \frac{15}{76} = 0.1974.$$

From Table IV:

$\frac{h_1}{\sigma}$	% area
0.56	0.21226
0.55	0.20884

Therefore $\frac{h_1}{\sigma} = 0.555.$

Similarly $\frac{h_2}{\sigma} = 0.517;$

$$\frac{h_1 + h_2}{\sigma} = 1.072 = \frac{2.22 - 1.78}{\sigma},$$

$$\therefore \quad \sigma = \frac{0.44}{1.072} = 0.4105;$$

$$\frac{h_1}{0.4105} = 0.555; \qquad \frac{h_2}{0.4105} = 0.517;$$

$$h_1 = 0.2278, \qquad h_2 = 0.2122.$$

Mean is at $1.78 + 0.2278 = 2.01$.

CHAPTER III

The Classes of Frequency Polygons

The plotted curve may fall into one of the following classes:

A. Unimodal.
  I. Simple.
    1. Range unlimited in both directions:
      a. Symmetrical. The normal curve.
      b. Unsymmetrical (Pearson's Type IV).
    2. Range limited in one direction, together with skewness (Types III, V and VI).
    3. Range limited in both directions:
      a. Symmetrical, Type II.
      b. Unsymmetrical, Type I.
  II. Complex.

B. Multimodal.

Classification. The classification of any given curve is not always an easy task. Whether the curve is unimodal or multimodal can be told by inspection. Whether any unimodal curve is simple or complex can not be told by any existing methods without great labor and uncertainty in the result.

Complex curves may be classified as follows:

1. Composed of two curves, whose modes are different but so near that the component curves blend into one; such curves are usually unsymmetrical.
2. The sum of two curves having the same mode but differing variability.
3. The difference of two curves having the same mode but differing variability.

If the material is believed to be *homogeneous* and the curve is unimodal it is probably *simple* and its classification may be carried further.

For classification arrange the classes as for finding mean square deviation (p. 30); also the mean cubed deviation and the mean of the deviations to the fourth power. Or in detail proceed as follows:

Having arranged the classes and corresponding frequencies in two adjacent columns, take a class near the mean (call it V_0) as a zero point; then the departure of all the other classes will be: -1, -2, -3, etc., and $+1$, $+2$, $+3$, etc.

Add the products of all these departures multiplied by the frequency of the corresponding class and divide by N; call the quotient ν_1.

Add the products of the *squares* of all the departures multiplied by the frequency of the corresponding class and divide by N; call the quotient ν_2.

Add the products of the *cubes* of all the departures multiplied by the frequency of the corresponding class and divide by N; call the quotient ν_3.

Add the products of the *fourth powers* of all the departures multiplied by the frequency of the corresponding class and divide by N; call the quotient ν_4. Or,

$$\nu_1 = \frac{\Sigma f(V - V_0)}{N} = \text{departure of } V_0 \text{ from mean. } V_0 \text{ being known, } M \text{ may be found } [M = V_0 + \nu_1];^*$$

$$\nu_2 = \frac{\Sigma f(V - V_0)^2}{N};$$

$$\nu_3 = \frac{\Sigma f(V - V_0)^3}{N};$$

$$\nu_4 = \frac{\Sigma f(V - V_0)^4}{N}.$$

The values ν_1, ν_2, ν_3, ν_4, are called respectively the first, second, third and fourth moments of the curve about V.

To get the moments of the curve about the mean, either of two methods (A or B) will be employed. Method A is used when integral variates are under consideration; method B when we deal with graduated variates.

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* This is the short method of finding M referred to on page 25.

(A) To find moments in case of integral variates:

$$\mu_1 = 0;$$

$$\mu_2 = \nu_2 - \nu_1^2;$$

$$\mu_3 = \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3;$$

$$\mu_4 = \nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4;$$

$$\mu_5 = \nu_5 - 5\nu_1\nu_4 + 10\nu_1^2\nu_3 - 10\nu_1^3\nu_2 + 4\nu_1^5;$$

$$\mu_6 = \nu_6 - 6\nu_1\nu_5 + 15\nu_1^2\nu_4 - 20\nu_1^3\nu_3 + 15\nu_1^4\nu_2 - 5\nu_1^6.$$

(B) To find moments in case of graduated variates:

$$\mu_1' = 0;$$

$$\mu_2' = \nu_2 - \nu_1^2 - \tfrac{1}{12};$$

$$\mu_3' = \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3;$$

$$\mu_4' = \nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4 - \tfrac{1}{2}(\nu_2 - \nu_1^2) + \tfrac{7}{240}.$$

Also, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$; $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$.

The probable errors of the preceding constants are as follows:

$$PE_{\mu_2} = 0.67449\sqrt{\frac{\mu_4 - \mu_2^2}{N}};$$

$$PE_{\mu_3} = 0.67449\sqrt{\frac{\mu_6 - \mu_3^2 - 6\mu_4\mu_2 + 9\mu_2^2}{N}};$$

In the special case of the normal curve the above probable errors become

$$PE_{\mu_2} = 0.67449\sigma^2\sqrt{\frac{2}{N}}; \quad PE_{\beta_2} = 0.67449\sqrt{\frac{24}{N}};$$

$$PE_{\mu_3} = 0.67449\sigma^3\sqrt{\frac{6}{N}}; \quad PE_{\sqrt{\beta_1}} = 0.67449\sqrt{\frac{6}{N}};$$

$$PE_{\mu_4} = 0.67449\sigma^4\sqrt{\frac{96}{N}}.$$

In the case of skew curves:

$$PE_D = 0.67449\sqrt{\frac{3}{2N}}\,\sigma \text{ (p. 53)};$$

$$PE \text{ of skewness} = 0.67449\sqrt{\frac{3}{2N}}. \quad \text{(See p. 46.)}$$

(From Pearson, 1903.)

The standard errors are derived from the corresponding probable errors by dividing by 0.67449.

The classification of any empirical frequency polygon depends upon the value of its "critical function," F* (Pearson, 1901).

$$F = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)}.$$

Value of F	Corresponding frequency curve
$F = \infty$	Type III. Transitional between Type I and Type VI
$F > 1$ and $< \infty$	Type VI
$F = 1$	Type V. Transitional between Type IV and Type II
$F > 0$ and < 1	Type IV
$F = 0,\ \beta_1 = 0,\ \beta_2 = 3$	Normal curve
$F = 0,\ \beta_1 = 0,\ \beta_2$ not $= 3$	Type II
$F < 0$	Type I

An important relation to be referred to later is

$$s = \frac{6(\beta_2 - \beta_1 - 1)}{3\beta_1 - 2\beta_2 + 6}.$$

---

* This value of F is general. For the special case of Types I–IV the following critical function was given by Pearson and has been much used. $F_1 = 2\beta_2 - 3\beta_1 - 6$. The classification was given as follows:

When F_1 is negative and $\begin{cases} \beta_1 > 0, \text{ curve is of Type I.} \\ \beta_1 = 0,\ \beta_2 < 3, \text{ curve is of Type II.} \end{cases}$

When $F_1 = 0$ and $\begin{cases} \beta_1 > 0, \beta_2 > 3, \text{ curve is of Type III.} \\ \beta_1 = 0,\ \beta_2 = 3, \text{ curve is normal.} \end{cases}$

When F_1 is positive and $\beta_1 > 0,\ \beta_2 > 3$, curve is of Type IV.

THE NORMAL FREQUENCY CURVE

The normal frequency curve (Fig. 14), obtained by rectangles or loaded ordinates, is symmetrical about a vertical (ordinate) erected perpendicular to the base at its middle point. This ordinate at the middle is also the highest one. The upper portion of the curve is concave toward the middle of the figure; the lower portions are concave toward the middle; and there is a point of inflection where the curvature changes. At the extremes the area of the curve is very small and, finally, with a finite number of observations, fades away to nothing.* If the whole area of the curve be assumed to be

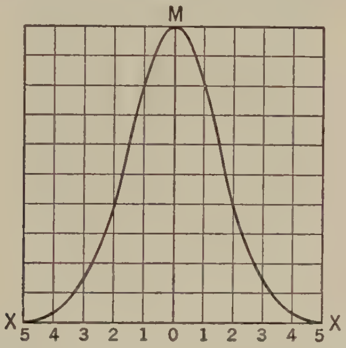

FIG. 14.—The normal frequency curve.

100 or 10,000 the proportion of the area of the curve between the ordinate at the middle abscissa and an ordinate at the abscissa representing any class or deviation removed at any distance from the ordinate at the middle abscissa or class, is called the probability integral. Table IV is a table of probability integrals computed for half a frequency curve.

The mathematical formula of the normal curve, a formula of which one does not have to understand the development in order to make use of it, is

$$y = \frac{N}{\sigma\sqrt{2\pi}} \cdot \frac{1}{e^{x^2/2\sigma^2}}.$$

* The theoretical range of this curve is infinite.

This formula gives the value of any ordinate y (or any class) at any distance x (measure along the base, X, X', of Fig. 14) from the mode. e is a constant number, 2.71828, the base of the Napierian system of logarithms. N is the total area of the curve or number of variates, and σ is the standard deviation, which is constant for any curve and measures the variability of the curve, or the steepness of its slope.

To compare any observed curve with the theoretical normal curve we can make use of tables. For the case of a polygon of loaded ordinates the theoretical frequency of any class at a deviation $\frac{x}{\sigma}$ from the mean can be taken directly from Table III. Here $\frac{x}{\sigma}$ is the actual deviation from the mean expressed in units of the standard deviation, and $\frac{y}{y_0}$ the corresponding ordinate, y_0 being taken as equal to 1, and σ is the standard deviation.

For the case of a polygon built up of rectangles representing the relative frequency of the variates, Table IV gives immediately the theoretical number of individuals occurring between the values $x = 0$ and $x = \pm\frac{x}{\sigma}$. By looking up the given values of $\frac{x}{\sigma}$ the corresponding theoretical percentage of variates between the limits $x = 0$ and $x = \pm\frac{x}{\sigma}$ will be found directly. The ratio $\frac{x}{\sigma}$ may be called the *index of abmodality.*

The normal curve may preferably be employed even when β_1 is not exactly equal to 0, nor β_2 exactly equal to 3, nor F exactly equal to 0. Use the normal curve when

$$F \times \mu_2^3 < \pm 1 \quad \text{and} \quad \frac{3\nu_2^2 - 2\nu_1^4}{\nu_4} = 1 \pm \cdot 2;$$

also the skewness (p. 53) should be less than twice the value $0.67449\sqrt{\frac{3}{2N}}$.

To Determine the Closeness of Fit ("Goodness of Fit") of a Theoretical Distribution to the Observed Distribution. The **Lexian ratio** L. tests the extent to which a particular set

of observations conforms to the normal frequency distribution (Wilhelm Lexis, 1877). This is the ratio between the σ of an observed distribution in a statistical series and the theoretical value of the standard deviation, $\sigma = \sqrt{Npq}$. The theoretical value is determined by the hypothesis that is being tested, as, for example, the Mendelian expectation or the Poisson series or the binomial distribution or other, similar series.

The application of the ratio may be illustrated by dice throwing. For each die, the probability (p) of a 6 turning up is 1 to 6, i.e., $p = \frac{1}{6}$; the probability (q) that something other than 6 will turn up is 5 to 6, i.e., $q = \frac{5}{6}$ (always $p + q = 1$). Twelve dice were thrown 4096 times, and the appearance of a 6 counted a success. Thus, for each throw, there might turn up 12 sixes, 11 sixes, 10, 9, 8, etc., to 0 sixes. Since the mean of a binomial distribution $= Np$ and the $\sigma = \sqrt{Npq}$, in the theoretical distribution of the number of 6's in the 12 dice thrown any number of times, the mean would be $12 \times \frac{1}{6} = 2$; the σ would be $\sqrt{12 \times \frac{1}{6} \times \frac{5}{6}} = 1.291$. For the actual trial, the results were as follows:

Successes	Frequency	Successes	Frequency
0	447	7	7
1	1145	8	1
2	1181	9	..
3	796	10	..
4	380	11	..
5	115	12	..
6	24	Total	4096

The mean of the distribution is 2.000, exactly as predicted. The σ is 1.296, or 0.005 larger than predicted. Since the Lexian ratio is the ratio between the observed σ and the theoretical σ, it equals $\frac{1.296}{1.291} = 1.0039$. Since this ratio will be 1.0 if the actual distribution has the same σ as the normal curve, the approach of the ratio to 1.0 shows how closely a given distribution approaches the theoretical normal curve.

The χ^2 (*"chi square"*) *test*. Find for each class the difference (δ_1) between the theoretical value (y) and the observed fre-

quency (f). Divide the square of this difference in each case by y. The sum of the quotients is χ^2, or

$$\chi^2 = \sum \frac{\delta_1{}^2}{y}$$

The probability ($P : 1$) that the observed distribution is truly represented by the theoretical polygon may be looked up in Table XII.

The value of P has been computed to 6 decimal places by Elderton. His table is published in Biometrika, I : 155–163, and is also reprinted in Pearson's "Tables for Statisticians and Biometricians." A new and more convenient table by R. A. Fisher (1925) is, by the courtesy of Fisher and his publishers, reproduced in our Table XII. "Instead of giving the values of P corresponding to an arbitrary series of values of χ^2, we have given," says Fisher, "the values of χ^2 corresponding to specially selected values of P."

In using Table XII it is necessary to know the value of N. The rule for finding N is that N "is equal to the number of degrees of freedom in which the observed series may differ from the hypothetical; in other words it is equal to the number of classes the frequencies in which may be filled up arbitrarily." This is at least one less than the actual number of classes that are being compared.

The **probable range** of abscissæ ($2\chi_l$) of a normal distribution, or that beyond which the theoretical frequency (y) is less than 1, varies with the number of variates (N) as well as with σ, in accordance with the following formula derived by the transposition of $y = \dfrac{N}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ by putting $y = 1$:

$$2x_l = 2\sigma \sqrt{\frac{2}{\log e} \log \frac{N}{\sigma\sqrt{2\pi}}}\,.$$

Example. For the ventricosity of 1000 shells of *Littornea littorea* from Tenby, Wales, $M = 90.964$ per cent and $\sigma = 2.3775$ per cent. What is the probable range of ventricosity expressed in per cent?

$$2x_e = 2 \times 2.3775 \sqrt{0.460517 \times \log \frac{1000}{2.506628 \times 2.3775}} = 15.2.$$

The observed range was 15 (Duncker, 1898). See also the criterion of Chauvenet (1888) for the rejection of extreme variates (p. 21).

The Normal Curve of Frequency as a Binomial Curve. The normal curve may also be expressed by the binomial formula $(p \times q)^\lambda$, where $p = \frac{1}{2}$, $q = \frac{1}{2}$ and λ is the number of terms, less 1, in the expansion of the binomial—hence approximately the number of classes into which the magnitudes of the variates should fall. If the standard deviation be known, λ may be found by the equation

$$\lambda = 4 \times (\text{standard deviation})^2 = 4\sigma^2.$$

Example of Normal Curve.

Number of rays in lower valve of *Pecten opercularis* from Firth of Forth:

V	f	$V - V_0$	$f(V - V_0)$	$f(V - V_0)^2$	$f(V - V_0)^3$	$f(V - V_0)^4$
14	1	−3	−3	9	−27	81
15	8	−2	−16	32	−64	128
16	63	−1	−63	63	−63	−63
17	154	0	0	0	0	0
18	164	1	164	164	164	164
19	96	2	192	384	768	1536
20	20	3	60	180	540	1620
21	2	4	8	32	128	512
N =	508		342	864	1446	4104

$$\nu_1 = \tfrac{342}{508} = 0.6732; \quad \nu_2 = \tfrac{864}{508} = 1.7008;$$

$$\nu_3 = \tfrac{1446}{508} = 2.8465; \quad \nu_4 = \tfrac{4104}{508} = 8.0787.$$

$M = V_0 + \nu_1 = 17 + 0.6732 = 17.6732.$

$\mu_2 = 1.7008 - 0.6732^2 = 1.2476; \quad \sigma = \sqrt{\mu_2} = 1.1170;$

$\mu_3 = 2.8465 - 3 \times 1.7008 \times 0.6732 + 2 \times 0.6732^3 = 0.0218;$

$\mu_4 = 8.0784 - 4 \times 2.8465 \times 0.6732 + 6 \times 1.7008 \times 0.6732^2$
$\qquad - 3 \times 0.6732^4 = 4.4220.$

$$\beta_1 = \frac{0.0218^2}{1.2476^3} = 0.0002; \quad \beta_2 = \frac{4.4220}{1.2476^2} = 2.8410.$$

$$F = \frac{0.0002(2.8410 + 3)^2}{4(4 \times 2.8410 - 3 \times 0.0002)(2 \times 2.8410 - 3 \times 0.0002 - 6)}$$

$= -0.00047. \qquad F\mu_2^3 - 0.0009.$

$$\frac{3\nu_2^2 - 2\nu_1^4}{\nu_4} = \frac{3(1.7008^2) - 2 \times 0.6732^4}{8.0787} = 1.0234.$$

Theoretical maximum frequency, $y_0 = \dfrac{N}{\sigma\sqrt{2\pi}} = \dfrac{508}{1.1170\sqrt{2\pi}} = 181.4.$

To find the average difference between the pth and the $(p+1)$th individual in any seriation (Galton's difference problem). Let i_p be the average interval between the pth and $(p+1)$th individual; n the total number of variates (by exception, instead of N); and σ their standard deviation. Then,

(1) when n is large and p small:

$$i_p = \sigma\frac{\sqrt{2\pi p}\ \ p^p e^{-p}}{\underline{|p}}\cdot\frac{1}{ny_m}\{1+c_1+c_2+c_3+\ldots\},$$

where $y_m = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}m^2}$.

m can be found from Table IV by the use of the formula

$$\frac{n-2p}{n} = \sqrt{\frac{2}{\pi}}\int_0^m e^{-\frac{1}{2}\chi^2}\,d\chi,$$

where the value of m sought is the argument corresponding to the tabular entry $\left(\dfrac{n-2p}{n}\right)$.

$$c_1 = 0.833\frac{(n-2p)^2}{n(n-p)p} - 2.5\frac{n-2p}{n^2}\frac{m}{y_m} + 1.875\frac{(n-p)p}{n^3}\cdot\left(\frac{m}{y_m}\right)^2;$$

$$c_2 = -0.75\frac{(n-p)^3+p^3}{n^2(n-p)p} + 1.5\frac{n-2p}{n^2}\frac{m}{y_m} - 0.125\frac{(n-p)p}{n^3}\left(7-\frac{4}{m^2}\right)\left(\frac{m}{y_m}\right)^2;$$

$$c_3 = -2.5\frac{(n-p)^5+p^5}{n^3(n-p)^2p^2} + 7.5\frac{(n-p)^4-p^4}{n^4(n-p)p}\cdot\frac{m}{y_m} - .625\frac{(n-p)^3+p^3}{n^5}\left(13-\frac{4}{m^2}\right)\left(\frac{m}{y_m}\right)^2 + .625\frac{(n-2p)(n-p)p}{n^5}\left(6-\frac{7}{m^2}\right)\left(\frac{m}{y_m}\right)^3 - .02083\left[\frac{(n-p)p^2}{n^3}\right]\frac{31m^4-101m^2+28}{y^4_m}.$$

The solution of the equations for c_1, c_2, and c_3 will be facilitated by finding, once for all, the logarithms of n, $(n-p)$, $(n-2p)$, $(n-p)p$, and $\frac{m}{y_m}$.

(2) When n and p are both large and not nearly equal:

$$i_p = \frac{\sigma}{ny_m}(1 + c_1 + c_2 + c_3 + \ldots).$$

(3) When n is small the unsimplified form of the equation must be used.

$$i_p = \sigma \frac{\underline{|n}}{\underline{|n-p}\,\underline{|p}} \frac{(n-p)^{n-p}p^p}{n^n} \sqrt{2\pi} \sqrt{\frac{(n-p)p}{n^3}} \times \frac{1}{y_m}(1 + c_1 + c_2 + c_3 + \ldots)$$

$\underline{|n}$ means the products of all integers from 1 to n. The series c_1, c_2, c_3 is not complete, but the values of c with higher subscripts are so small that they may be neglected.

Let $I_{p'p''}$ be the difference measured in units of σ between the p'th and the p''th individual, then

$$I_{p'p''} = (i_{p'} + i_{p'+1} + i_{p'+2} + \ldots + i_{p''-1})\sigma.$$

The foregoing method is that of Pearson (1902[b]) based upon some considerations of Galton (1902).

To find the best fitting normal frequency distribution when only a portion of an empirical distribution is given. First apply the following parabola of the second order:

$$(1) \qquad y = y_0\left\{\epsilon_0 + \epsilon_1\frac{x}{l} + \epsilon_2\left(\frac{x}{l}\right)^2\right\},$$

where l is the half range and

$$\epsilon_0 = \tfrac{3}{4}(3\lambda_0 - 5\lambda_2) = 3(\lambda_2 - 0.2\epsilon_2);$$

$$\epsilon_1 = 3\lambda_1;$$

$$\epsilon_2 = 3.75(3\lambda_2 - \lambda_0);$$

also,

$$y_0 = \frac{m_0}{2l}; \quad \lambda_0 = 3\lambda_2 - \frac{4}{15}\epsilon_2; \quad \lambda_1 = \frac{m_1}{m_0 l}; \quad \lambda_2 = \frac{m_2}{m_0 l^2}.$$

To find m_0 arrange the frequencies in the usual manner (p. 30) and find the logarithm of each; their sum is equal to m_0. Making the class situated at the middle of the range 0, find the deviation of each of the other classes from this class. The algebraic sum of the product of the logarithms by the deviations gives m_1. The second moment about the same zero point gives m_2. Or,

$$m_0 = \Sigma \log f = \Sigma Y; \quad m_1 = \Sigma[Y(V-V_0)]; \quad m_2 = \Sigma[Y(V-V_0)^2].$$

Substituting in (1) we get a numerical quadratic equation which can be put in the form

$$(2)\quad Y = y_0 \left\{ \epsilon_2 \left[\left(\frac{x}{l}\right)^2 + \frac{\epsilon_1}{\epsilon_2}\frac{x}{l} + \left(\frac{\epsilon_1}{2\epsilon_2}\right)^2 \right] + \epsilon_0 - \epsilon_2 \left(\frac{\epsilon_1}{2\epsilon_2}\right)^2 \right\}$$

$$= y_0 \left\{ \epsilon_2 \left(\frac{x + \frac{\epsilon_1 l}{2\epsilon_2}}{l} \right)^2 + \epsilon_0 - \frac{\epsilon_1^2}{4\epsilon_2} \right\}.$$

If the normal curve be $y = z_0 e^{-\frac{(x+h)^2}{2\sigma^2}}$,

$$(3)\qquad Y = \log y = \log z_0 - \frac{(x+h)^2}{2\sigma^2} \log e;$$

whence, by comparison of right-hand expressions in equations (2) and (3),

$$\log z_0 = y_0 \times \left(\epsilon_0 - \frac{\epsilon_1^2}{4\epsilon_2} \right);$$

$$2\sigma^2 = \frac{l^2 \log \epsilon}{y_0 \times \epsilon_2}.$$

Then the required normal curve is

$$y = z_0 e^{-x^2/2\sigma^2}.$$

(Pearson, 1902[c].)

Other Unimodal Frequency Polygons. The formulas of Pearson's Types I to VI are as follows:

Type I. $\quad y = y_0 \left(1 + \frac{x}{l_1}\right)^{m_1} \left(1 - \frac{x}{l_2}\right)^{m_2}.$

Type II. $y = y_0 \left(1 - \frac{x^2}{l^2}\right)^m$.

Type III. $y = y_0 \left(1 + \frac{x}{l}\right)^p e^{-x/d}$.

Type IV. $y = y_0 \cos \theta^{2m} e^{-r\theta}$, where $\tan \theta = \frac{x}{l}$.

Type V. $y = y_0 x^{-p} e^{-\gamma/x}$.

Type VI. $y = y_0 (x - l)^{q_2} / x^{q_1}$.

In these formulas:

x, abscissæ;

y_0, the ordinate at the origin, to be especially reckoned for each type;

y, the height of the ordinate (or rectangle) located at the distance x from y_0;

l, a part of the abscissa axis XX' expressed in units of the classes;

e, the base of the Napierian system of logarithms, 2.71828.

The other letters stand for relations that are explained in the sections below treating of each type separately.

The **range of the curve** is limited in both directions in Types I and II, is limited in one direction only in Types III, V and VI, and is unlimited in both directions in Type IV and the normal curve. The normal curve may give the best fit, however, notwithstanding the fact that in biological statistics the range is ordinarily limited at both extremes. Thus the range of carapace length to total length of the lobster is limited between 0 and 1. The ratio of carapace length to abdominal length in various crustaceans may, however, conceivably take any value from $+\infty$ to 0. In the ratio of dorsoventral to antero-posterior diameter the forms of the molluscan genera Pinna or Malleus on the one hand and Solen on the other approach such extremes.

Asymmetry or Skewness (α) is found in Types I, III, IV, V and VI. In skew curves the mode and the mean are separated from each other by a certain distance D; or $D =$ mean − mode. Asymmetry is measured by the ratio $\alpha = \frac{D}{\sigma}$. If the mean is greater than the mode, skewness is positive; if the mean is less than the mode, skewness is negative. D,

and hence skewness, may be calculated when the theoretical mode is known (see p. 28, and below).

In Types I and IV skewness is measured also by the ratio $\alpha = \frac{1}{2}\sqrt{\beta_1}\frac{s \pm 2}{s \mp 2}$, where $s = \frac{6(\beta_2 - \beta_1 - 1)}{3\beta_1 - 2\beta_2 + 6}$. If $5\beta_2 - 6\beta_1 - 9$ is positive, α has the sign of μ_3; if negative, α has the opposite sign to μ_3 (Duncker, 1900).

In Type I, $\alpha = \frac{1}{2}\sqrt{\beta_1}\frac{s+2}{s-2}\left(= \frac{1}{2}\sqrt{\beta_1}\frac{\beta_2 + 3}{5\beta_2 - 6\beta_1 - 9}\right)$

In Type III, $\alpha = \frac{1}{2}\sqrt{\beta_1} = \frac{\pm \mu_3}{+2\sqrt{\mu_2^3}}$, where the sign is the same as that of μ_3:

In Type IV, $\alpha = \frac{1}{2}\sqrt{\beta_1}\frac{s-2}{s+2}$,

In Type V, $\alpha = \frac{2\sqrt{p-3}}{p}$,

since $p - 4$ is the positive root of the quadratic:

$$(p-4)^2 - \frac{16}{\beta_1}(p-4) - \frac{16}{\beta_1} = 0,$$

p is readily found.

In Type VI, $\alpha = \frac{(q_1 + q_2)\sqrt{(q_1 - q_2 - 3)}}{(q_1 - q_2)\sqrt{\{(q_1 - 1)(q_2 + 1)\}}}$,

where $(1 - q_1)$ and $(q_2 + 1)$ are the two roots of the equation

$$z^2 - sz + \frac{s^2}{4 + \frac{1}{4}\beta_1(s+2)^2/(s+1)} = 0.$$

To compare any observed frequency polygon of Type I with its corresponding theoretical curve.

$$y = y_0\left(1 + \frac{x}{l_1}\right)^{m_1}\left(1 - \frac{x}{l_2}\right)^{m_2}.$$

To find l_1, l_2, m_1, m_2, y_0.

The total range, l, of the curve (along the abscissa axis) is found by the equation

$$l = \frac{\sigma}{2}\sqrt{\beta_1(s+2)^2 + 16(s+1)};$$

l_1 and l_2 are the ranges to the one side and the other of y_0;

$$l_1 = \tfrac{1}{2}(l - Ds); \qquad D = \sigma\alpha = \sqrt{\mu_2}\cdot\alpha;$$

$$l_2 = l - l_1;$$

$$m_1 = \frac{l_1}{l}(s-2); \qquad m_1 + m_2 = s - 2;$$

$$y_0 = \frac{N}{l}\cdot\frac{m_1^{m_1}\cdot m_2^{m_2}}{(m_1+m_2)^{m_1+m_2}}\cdot\frac{\Gamma(m_1+m_2+2)}{\Gamma(m_1+1)\Gamma(m_2+1)}.$$

To solve this equation it will be necessary to determine the value of each parenthetical quantity following the Γ sign and find the corresponding value of Γ from Table XIX. It is, however, sometimes easier to calculate the value of y_0 from the following approximate formula:

$$y_0 = \frac{N}{l}\cdot\frac{(m_1+m_2+1)\sqrt{m_1+m_2}}{\sqrt{2\pi m_1 m_2}}\,e^{\frac{1}{12}\left(\frac{1}{m_1+m_2}-\frac{1}{m_1}-\frac{1}{m_2}\right)}.$$

With these data the theoretical curve of Type I may be drawn. Frequency polygons of Type I are often found in biological measurements.

To compare any observed frequency polygon of Type II with its corresponding theoretical curve.

$$y = y_0\left(1 - \frac{x^2}{\frac{1}{4}l^2}\right)^m.$$

This equation is only a special form of the equation of Type I in which $l_1 = l_2$ and $m_1 = m_2$.

As from page 44, $\beta_1 = 0$ in Type II, $l = 2\sigma\sqrt{s+1}$; since the curve is symmetrical, $D = 0$, and

$$m = \tfrac{1}{2}(s-2); \quad y_0 = \frac{N}{\frac{1}{2}l}\,\frac{\Gamma(m+1.5)}{\sqrt{\pi}\,\Gamma(m+1)}.$$

The Γ values will be found from Table XIX.

An approximate formula for y_0 is given by Duncker as follows:

$$y_0 = \frac{N}{\sigma\sqrt{2\pi}}\,\frac{s-1}{\sqrt{(s+1)(s-2)}}\,e^{-\frac{1}{4(s-2)}}.$$

To compare any observed frequency polygon of Type III with its corresponding theoretical curve.

$$y = y_0\left(1 + \frac{x}{l_1}\right)^p e^{-x/d}.$$

The range at one side of the mode is infinite; at the other is found by the formula

$$l_1 = \sigma\frac{4 - \beta_1}{2\sqrt{\beta_1}} = \sigma\frac{1 - \alpha^2}{\alpha} \text{ (for Type III).}$$

Also, $$p = \frac{l_1}{D} = \frac{l_1}{\sigma\alpha}; \quad y_0 = \frac{N}{l_1}\cdot\frac{p^{p+1}}{e^p\Gamma(p+1)}.$$

The value of Γ corresponding to $p + 1$ can be got from Table XIX.

To compare any observed frequency polygon of Type IV with its corresponding theoretical curve. This is the commonest type of biological skew curves.

$$y = y_0(\cos\theta)^{2m}\cdot e^{-\tau\theta}.$$

θ is a variable, dependent upon x as shown in the equation

$$x = u \tan\theta.$$

The factor $(\cos\theta)^{2m}$ following y_0 indicates that the curve is not calculated from the mean ordinate (A), or the mode ($A - D$), but that the zero ordinate is at $A - mD$; or at a distance $m \times D$ from the mean.

$$u = \frac{\sqrt{\mu_2}}{4}\sqrt{16(s-1) - \beta_1(s-2)^2}; \quad m = \tfrac{1}{2}(s+2);$$

$$D = \frac{\sigma}{2}\sqrt{\beta_1}\frac{s-2}{s+2}; \qquad mD = \frac{\sigma}{4}\sqrt{\beta_1}(s-2);$$

$$\tau = \frac{\sqrt{\mu_2}s(s-2)\sqrt{\beta_1}}{4u}, \text{ with the opposite sign to } \mu_3;$$

$$\theta \text{ (arc of circle)} = \frac{\pi\theta^\circ}{180^\circ};$$

$$y_0 = \frac{N}{u}\sqrt{\frac{s}{2\pi}}\,\frac{e^{\frac{(\cos\phi)^2}{3s} - \frac{1}{12s} - \tau\phi}}{(\cos\phi)^{s+1}} *$$

$$\phi = \text{angle whose tangent is } \frac{\tau}{s}.$$

To compare any observed frequency polygon of Type V with its corresponding theoretical curve.

$$y = y_0 x^{-p} e^{-\gamma/x}$$

To find p solve the quadratic equation

$$(p-4)^2 - \frac{16}{\beta_1}(p-4) - \frac{16}{\beta_1} = 0,$$

and take the positive root.

$$\left.\begin{array}{l}\gamma = \sigma(p-2)\sqrt{p-3} \\ \quad \text{with the sign of } \mu_3\end{array}\right\}; \quad y_0 = \frac{N\cdot\gamma^{p-1}}{\Gamma(p-1)}; \quad D = \frac{2\gamma}{p(p-2)}.$$

To compare any observed frequency polygon of Type VI with its corresponding theoretical curve.

$$y = y_0(x - l_1)^{q_2}/x^{q_1}.$$

$1 - q_1$ and $q_2 + 1$ are the two roots of the equation

$$z^2 - sz + \frac{s^2}{4 + \frac{1}{4}\beta_1(s+2)^2/(s+1)} = 0;$$

$$l_1 = s\sqrt{\frac{\mu_2(s+1)}{(1-q_1)(1+q_2)}},$$ where $(1 - q_1)$ and s are negative;

$$y_0 = \frac{n l_1^{q_1 - q_2 - 1}\Gamma(q_1)}{\Gamma(q_1 - q_2 - 1)\Gamma(q_2 + 1)}:$$

$$D = \frac{l(q_1 + q_2)}{(q_1 - q_2)(q_1 - q_2 - 2)}.$$

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* The foregoing value is approximate and is applicable when, as is usually the case, s is greater than 2. The exact value is given by Pearson as

$$y_0 = \frac{N}{u}\cdot\frac{e^{\frac{1}{2}\tau\pi}}{\int_0^\pi (\sin\theta)^s e^{\tau\theta}\,d\theta},$$

the formula for reducing which is to be gained from the integral calculus.

Example of Calculating the Theoretical Curve Corresponding with Observed Data (Fig. 12). Distribution of frequency of glands in the right foreleg of 2000 female swine (integral variates):

Number of glands	0	1	2	3	4	5	6	7	8	9	10
Frequency......	15	209	365	482	414	277	134	72	22	8	2

Assume the axis yy' (V_0) to pass through ordinate 4; then:

V	$V - V_0$	f	$f(V - V_0)$	$f(V - V_0)^2$	$f(V - V_0)^3$	$f(V - V_0)^4$
0	-4	15	-60	240	-960	3840
1	-3	209	-627	1881	-5643	16929
2	-2	365	-730	1460	-2920	5840
3	-1	482	-482	482	-482	482
4	0	414	0	0	0	0
5	1	277	277	277	277	277
6	2	134	268	536	1072	2144
7	3	72	216	648	1944	5832
8	4	22	88	352	1408	5632
9	5	8	40	200	1000	5000
10	6	2	12	72	432	2592
	Σ	2000	-998	6148	-3872	48568

$\nu_1 = -998 \div 2000 = -0.499;$

$\nu_2 = -6148 \div 2000 = 3.074;$

$\nu_3 = -3872 \div 2000 = -1.936;$

$\nu_4 = -48{,}568 \div 2000 = 24.284.$

$\mu_1 = M = 4 - 0.499 = 3.501;$

$\mu_2 = 3.074 - (-0.499)^2 = 2.824999;$

$\mu_3 = -1.936 - 3(-0.499 \times 3.074) + 2(-0.499)^3 = 2.417275;$

$$\mu_4 = 24.284 - 4(-0.499 \times -1.936) + 6(0.499^2 \times 3.074) - 3(0.499)^4 = 24.826314.$$

$$\beta_1 = \frac{(2.417275)^2}{(2.824999)^3} = 0.259177;$$

$$\beta_2 = \frac{24.826314}{(2.824999)^2} = 3.110825.$$

$$F = \frac{0.259177(3.110825 + 3)^2}{\begin{Bmatrix} 4(4 \times 3.110825 - 3 \times 0.259177) \\ (2 \times 3.110825 - 3 \times 0.259177 - 6) \end{Bmatrix}} = -0.373.$$

$\therefore$ Type I.

$$s = \frac{6(3.110825 - 0.259177 - 1)}{3 \times 0.259177 - 2 \times 3.110825 + 6} = 19.986091.$$

$$\alpha = \tfrac{1}{2}\sqrt{0.259177}\,\frac{3.110825 + 3}{5 \times 3.110825 - 6 \times 0.259177 - 9} = 0.311157.$$

$$D = \sqrt{2.824999} \times 0.311157 = 0.522984.$$

$$D \cdot s = 0.522984 \times 19.986091 = 10.452406.$$

$$l = \frac{1.680773}{2}\sqrt{0.259177(19.986091 + 2)^2 + 16(19.986091 + 1)}$$
$$= 18.045048.$$

$$l_1 = \frac{18.045048 - 10.452406}{2} = 3.796321.$$

$$l_2 = 18.045048 - 3.796321 = 14.248727.$$

$$m_1 = \frac{3.796321 \times 17.986091}{18.045048} = 3.783918.$$

$$m_2 = 17.986091 - 3.783918 = 14.202173.$$

$$y_0 = \frac{2000}{18.0450} \cdot \frac{18.9861 \times \sqrt{17.9861}}{\sqrt{2\pi \times 3.7839 \times 14.2021}} e^{1/12(.0556 - .2643 - .0704)}$$

$= 474.50$ the frequency of the modal class.

Position of the mode, $y_0 = A - D = 3.501 - 0.523 = 2.978.$ The closeness of fit to the theoretical curve is calculated below by Pearson's method (pp. 47, 48).

V	f	Theoretical (y)	δ	δ^2	$\frac{\delta^2}{y}$
−1	0	0	0	0	0
0	15	19.4	− 4.4	19.36	1.00
1	209	180.9	28.1	789.61	4.36
2	365	391.1	−26.1	681.21	1.74
3	482	474.5	7.5	56.25	0.12
4	414	408.9	5.1	26.01	0.06
5	277	274.7	2.3	5.29	0.02
6	134	149.6	−15.6	243.36	1.63
7	72	67.0	5.0	25.00	0.37
8	22	24.7	− 2.7	7.29	0.30
9		* 7.3	*		
10	10	1.7	0.7	0.49	0.05
11		0.3			

$$\Sigma \frac{\delta^2}{y} = 9.65 = \chi^2$$

degrees of freedom $= 10 - 3 = 7.$

Three degrees of freedom have been absorbed in finding the mean, standard deviation and adjusting the total frequency. Looking up the entry of 9.65 under $N' = 7$ in Table XII we find the $P = 0.21$. That is, the probability is that in 21 out of every 100 random series belonging to Type I we should expect a fit not essentially closer than that given by our series, which, of course, assures us that this distribution is properly classified under Type I.

* Classes with a frequency of less than 10 should be grouped together.

The Use of Logarithms in Curve-fitting. Most of the statistical operations can be greatly facilitated by the use of logarithms. In curve-fitting their use becomes necessary. The following paradigm will be found of assistance:

GENERAL

$\log \nu_1 = \log (V - V_0) - \log N. \qquad M = V_0 + \nu_1.$

$\log \nu_2 = \log (V - V_0)^2 - \log N. \quad \log \sigma = \frac{1}{2} \log \mu_2.$

$\log \nu_3 = \log (V - V_0)^3 - \log N. \quad \log C = \frac{1}{2} \log \mu_2 - \log M + 2.$

$\log \nu_4 = \log (V - V_0)^4 - \log N.$

$\log PE_M = 9.828982 + \log \sigma - \frac{1}{2} \log N.$

$\log PE_\sigma = \log PE_M - 0.150515.$

$\log PE_C = \log PE_\sigma - \log M + 2.$

$\log 2 = 0.301030$	$\frac{1}{12} = 0.0833333.$	Find: $2 \log \nu_1$.
$\log 3 = 0.477121$	$\frac{7}{240} = 0.0291667.$	$3 \log \nu_1$.
$\log 4 = 0.602060$	$\frac{3}{240} = 0.0125000.$	$4 \log \nu_1$.
$\log 6 = 0.778151$	$\log \frac{1}{2} = 9.698970.$	

$\mu_2 = \mathfrak{N} (\log \nu_2) - \mathfrak{N} (2 \log \nu_1) - 0.083333.$ Find: $\log \mu_2$; $2 \log \mu_2$; $3 \log \mu_2$.

$\mu_3 = \mathfrak{N} (\log \nu_3) - \mathfrak{N} (\log 3 + \log \nu_1 + \log \nu_2) + \mathfrak{N} (\log 2 + 3 \log \nu_1).$ Find: $\log \mu_3$; $2 \log \mu_3$.

$$\mu_4 = \mathfrak{N} (\log \nu_4) - \mathfrak{N} (\log 4 + \log \nu_1 + \log \nu_3) + \mathfrak{N} (\log 6 + 2 \log \nu_1 + \log \nu_2) - \mathfrak{N} (\log 3 + 4 \log \nu_1) - \mathfrak{N} (9.698970 + \log \mu_2) + \tfrac{17}{240}.$$ Find $\log \mu_4$.

$$\log \beta_1 = 2 \log \mu_3 - 3 \log \mu_2.$$

$$\log \beta_2 = \log \mu_4 - 2 \log \mu_2.$$

$$w = 5\beta_2 - 6\beta_1 - 9 \text{ (Types I, IV).}$$

Skewness:

Type I. $\log \alpha = \frac{1}{2} \log \beta_1 - \log w + \log (\beta_2 + 3) - 0.301030.$

Type III: $\log \alpha = \frac{1}{2} \log \beta_1 - 0.301030.$

Type IV: $\log \alpha = \frac{1}{2} \log \beta_1 - \log (\beta_2 + 3) + \log w - 0.301030.$

Type V: $\log \alpha = \log 2 + \frac{1}{2}\log (p - 3) - \log p.$

Type VI: $\log \alpha = \log (q_1 + q_2) + \frac{1}{2}\log (q_1 - q_2 - 3)$
$- \log (q_1 - q_2) - \frac{1}{2}\log (q_1 - 1)$
$- \frac{1}{2}\log (q_2 + 1).$

Type IV

This is the most difficult of all the types to be fitted. The work of fitting is carried out by the use of logarithms, as follows:

$\log j = \frac{1}{2}\log \beta_1 + \log (s - 2).\quad \log k = \log j + \frac{1}{2}\log \mu_2.$

$\log \alpha = \log j - \log (s + 2) - 0.301030.$

$$\log u = \frac{1}{2}\log \mu_2 + \frac{1}{2}\log \{\mathfrak{N}[\log (s - 1) + 1.204120] - \mathfrak{N}[\log \beta_1 + 2\log (s - 2)]\} - 0.602060.$$

$\log D = \log \alpha + \frac{1}{2}\log \mu_2;\quad m = \dfrac{s + 2}{2}.$

$\log mD = \log k - 0.602060.$

$\log \tau = \log k + \log s - 0.602060 - \log u.$

$\log \tan \phi = \log \tau - \log s.$

$\log \theta = 8.241877 + \log \theta^\circ.$*

$$\log y_0 = \log N + \frac{1}{2}\log s + \mathfrak{N}\{\log[\mathfrak{N}(2\log\cos\phi - \log 3s) - \mathfrak{N}(8.920819 - \log s) - \mathfrak{N}(\log\tau + \log\phi)] + 9.637784\} - 0.399090 - \log u - (s + 1)\log\cos\phi.$$

$$\log y = \log y_0 + \mathfrak{N}\log (s + 2) + \log\log\cos\theta - \mathfrak{N}[9.63778 + \log\theta^\circ{}^* + \log\tau].$$

Multimodal Curves

Multimodal curves are given when the frequency in the different classes exhibits more than one mode. False multimodal curves result from too few observations, or when the classes are too numerous for the variates. By increasing the number of variates or by making the classes more inclusive some of the modes disappear.

* In degrees and fractions of a degree.

Multimodal curves differ in degree. The modes may be so close that only a single mode (usually in an asymmetrical

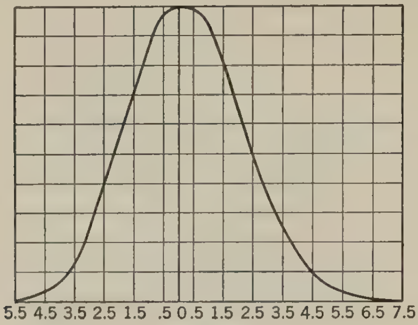

FIG. 15

curve) appears in the result; or one of the modes may appear as a hump on the other; or the two modes may even be far apart and separated by a deep sinus (Figs. 15 to 18).

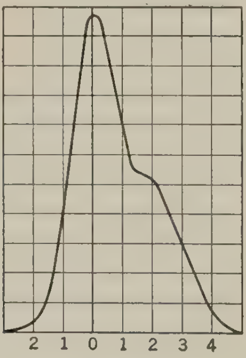

FIG. 16.

Pearson has offered a means of breaking up a compound curve with apparently only one mode into two curves having distinct modes; but this method is very tedious and rarely applicable.

The **index of divergence** of two modes of a multimodal curve is the distance between the modes expressed in terms

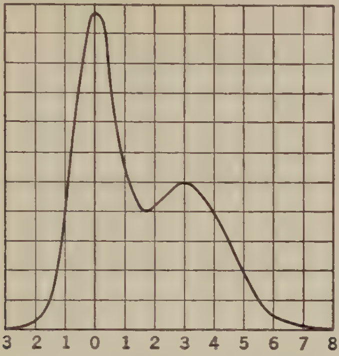

FIG. 17

of the standard deviation of the more variable of the components.*

The **index of isolation** of two masses of variates grouped

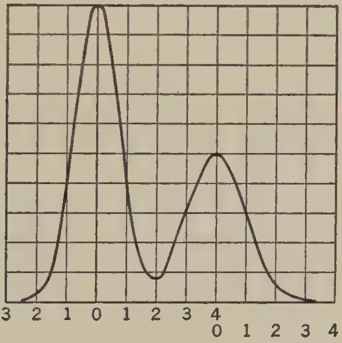

FIG. 18

about adjacent modes is the ratio of the depression between the modes to the height of the shorter mode.

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* I have proposed (Science, VII, 685) to measure the divergence in a unit = 3 × standard deviation, which has certain advantages in species study. C. B. D.

The meaning of multimodal curves is diverse. Sometimes they indicate a polymorphic condition of the species, the modes representing the different type forms. This is the case with the number of ray flowers of the white daisy which has modes at 8, 13, 21, 34, etc. Sometimes they indicate a splitting of a species into two or more varieties.

See also Chapter IV.

CHAPTER IV

Analysis of Variance

General

Variance, as the measure of the degree of variability, is the mean of the squared deviations from the mean, or the second moment about the mean. It is equal to the square of the standard deviation.

It frequently happens that the variants do not all belong to the same homogeneous group. The total variability can often, in such cases, be analyzed to advantage. And it is a remarkable property of the variance of a variable that, when it has been analyzed under several independent causes, the total variance is the sum of the variance due to each cause.

To analyze variance, arrange the variants in a table, placing, in one row, the frequencies of one class, distributed with reference to the second varying or differentiating factor; and so proceed with the various rows. This class of cases is illustrated in the example (Table 2), in which is given the number of well-developed fruits in various (10) flower clusters taken at random from each of ten dogwood trees (a–j).

Starting with such a distribution we may seek an answer to the question in how far is the collection of data homogeneous? There are differences, clearly enough, between the averages of the different trees. What is the probability that such differences are due merely to random sampling?

In the analysis of variance from this point there are two methods according as the total frequencies of the columns are all equal or, on the other hand, unequal.

1. Numbers of the Observations in Each Class Are Equal

The deviation of any class entry (such as the number of fruits in a head from any one tree) from the grand mean of all

fruits from all trees is the deviation of such entry from the mean of such class plus the deviation of the class mean from the grand mean: or, $x - M_x = (x - M_s) + (M_s - M_x)$; where x is the individual variant, M_x the mean of all the variants, or the grand mean, and M_s is the mean of the class (in the example, the sample derived from one tree).

The total sum of squares is obtained by squaring the above equation as many times as there are observations, and then summing up all these squared deviations. This result may be expressed as follows:

total sum of squares

$$= \Sigma(x - M_x)^2 = \Sigma_s\Sigma(x - M_s)^2 + n\Sigma_s(M_s - M_x)^2$$

where n is the frequency in each group, and Σ_s indicates that the summation is made for all (s) groups.

If the data are grouped and one assumes a centrally placed guessed mean the arithmetical labor of computing the deviation is much reduced. The various components of the above equation may then be found as follows:

total sum of squares

$$= \Sigma(x - M_x)^2 = \Sigma(x')^2 - N(M'_x)^2;$$

sum of squares within classes

$$= \Sigma_s\Sigma(x - M_s)^2 = \Sigma(x')^2 - n\Sigma_s(M'_s)^2;$$

sum of squares between classes

$$= n\Sigma_s(M_s - M_x)^2 = n\Sigma_s(M'_s)^2 - N(M'_x)^2;$$

where the prime (′) indicates that the symbol it accompanies is measured from an arbitrary origin.

N is the total number of observations.

n is the number of observations within each class.

Example (Table 2).

$$\Sigma(x - M_x)^2 = 738 - 77.44 = 660.56$$

$$\Sigma_s\Sigma(x - M_s)^2 = 738 - 456 = 282.00$$

$$n\Sigma_s(M_s - M_x)^2 = 456 - 77.44 = 378.56$$

TABLE 2.—DISTRIBUTIONS OF NUMBERS OF WELL-DEVELOPED FRUITS PER CLUSTER OF DIFFERENT DOGWOOD TREES (*Cornus floridum*) LOCATED AT COLD SPRING HARBOR

Number of fruits per cluster	Plant										Total
	a	b	c	d	e	f	g	h	i	j	
1	3	3	2	2	1				5		16
2	3	3	3	4	1	2	1		1		18
3	4	1	1	2	1	3			3		15
4		3	1	1	2	4	2		1		14
5			1		1		1			2	5
6			1		1	1	3	4		2	12
7				1	3		1	2		1	8
8			1				1	3		3	8
9										2	2
10											0
11											0
12							1	1			2
Total . . .	10	10	10	10	10	10	10	10	10	10	100
$\Sigma d'$	−29	−26	−16	−23	−4	−15	10	24	−30	21	−88
$\frac{\Sigma d'}{n} = \nu_1$	−2.9	−2.6	−1.6	−2.3	−0.4	−1.5	1.0	2.4	−3.0	2.1	−0.88
$\left(\frac{\Sigma d'}{n}\right)^2$	8.41	6.76	2.56	5.29	0.16	2.25	1.00	5.76	9.00	4.41	0.7744

2. Numbers of Observations in Each Class Are Unequal

The general method follows that of equal classes, but with some restriction. The general equation for finding the sum of squares when the size of classes is equal (p. 66) is to be modified as follows:

$$\Sigma(x - M_x)^2 = \Sigma_s\Sigma(x - M_s)^2 + \Sigma_s n_s(M_s - M_x)^2.$$

If the data are grouped and if one assumes a centrally placed guessed mean the procedure for finding the various sum of squares is as follows:

total sum of squares

$$\Sigma(x - M_x)^2 = \Sigma(x')^2 - N(M'_x)^2;$$

sum of squares within classes

$$= \Sigma_s\Sigma(x - M_s)^2 = \Sigma(x')^2 - \Sigma n_s(M'_s)^2;$$

sum of squares between classes

$$= \Sigma_s n_s(M_s - M_x)^2 = \Sigma_s n_s(M'_s)^2 - N(M'_x)^2.$$

Example (Table 3).

$$\Sigma(x - M_x)^2 = 830 - 12.35 = 817.65.$$

$$\Sigma_s\Sigma(x - M_s)^2 = 830 - 168.92 = 661.08.$$

$$\Sigma_s n_s(M_s - M_x)^2 = 168.92 - 12.35 = 156.87.$$

To get the mean square between different classes (e.g., trees) divide by the number of "degrees of freedom," which is one less than the number of samples. This procedure is justified on the ground that in a series of n deviations from the mean (of which the sum is zero) the nth deviation is fixed and should not be regarded as an independent contributor to the mean variance. The same procedure is followed to get the mean square variation within classes.

The total number of observations minus 1 gives the total number of "degrees of freedom." This can be analyzed into two parts: (1) the degrees of freedom between classes, and (2) the degrees of freedom within classes.

The results of Analyses 1 and 2 can be best brought out by

TABLE 3.—DISTRIBUTIONS OF THE NUMBER OF LEAFLETS FOUND ON THE VARIOUS LEAVES OF 15 EXAMPLES OF THE SUMAC (*Rhus glabra*), AT COLD SPRING HARBOR

Number of leaflets	Number of individual plant															Total
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
11–12									1	1				2		4
13–14	1								2	3			1	2	1	10
15–16	2		4	2	2		14	2	4	2	3		2	1	3	41
17–18	1	2	5	7	16	6	18	6	3	2	12	14	3	2	8	105
19–20	2	3	2	10	16	6	1	2	1	1	20	15	2	2	3	86
21–22	2	2	1	1	1	1				3	4	6	2	2	6	31
23–24	3	1							3	1	2	6	6	3	10	35
25–26		4							3	3				1	1	12
27–28		1								2			5			8
n	11	13	12	20	35	13	33	10	17	18	41	41	21	15	32	332
$\Sigma d'$	0	18	−12	−10	−19	−5	−46	−10	−6	−3	−10	4	24	−7	12	−64
$\frac{\Sigma d'}{n} = \nu_1$	0	1.385	−1.000	−0.500	−0.543	−0.385	−1.394	−1.000	−0.353	0.167	−0.244	0.098	1.306	−0.467	0.141	−0.193
$n\left(\frac{\Sigma d'}{n}\right)^2$	0	24.922	12.000	5.000	10.315	1.923	64.119	10.000	2.117	0.500	2.440	0.390	27.430	3.267	4.499	12.35

arranging the data as follows, and then by computing the mean square:

1. ANALYSIS OF VARIANCE, DOGWOOD FRUITS

Source of variation	Sum of squares	Degrees of freedom	Mean square
Variation within classes.......	282.00	90	3.13
Variation between classes......	378.56	9	42.06
Total..................	660.56	99	6.67

2. ANALYSIS OF VARIANCE, *R. glabra*

Source of variation	Sum of of squares	Degrees of freedom	Mean square
Variation within classes.......	661.08	317	2.09
Variation between classes......	156.67	14	11.18
Total..................	817.75	331	2.47

To test whether the mean square between classes is significantly different from the mean square within classes, it is necessary to compute

$$F = \frac{\text{larger mean square}}{\text{smaller mean square}}.$$

If F is as great as the light-faced figures given in Table XIII*, the difference between the mean squares is considered to be statistically significant, since only 5 times in 100 trials would random sampling give a difference of the magnitude. If F is as great as, or greater than, the dark-faced numbers the difference is considered very significant since only once, or less, in 100 trials would random sampling give a difference of this magnitude.

In Analysis 1, $F = \frac{42.06}{3.13} = 13.4$. Entering the table, it is seen that an F of about 2.72 is very significant, and therefore the variation between trees is greater than that within trees.

In Analysis 2, $F = \frac{11.18}{2.09} = 5.3$. The table shows that an F of about 2.24 is very significant, and, therefore, the variation between plants is greater than that within plants.

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* For descriptions of Table XIII see p. 152.

CHAPTER V

Correlated Variability and Measures of Relationship

Definition. When two or more traits, characters or sets of values show mutual or reciprocal correspondence of relationship in their variability, they are said to be *correlated*. Correlation has been variously defined as: (1) the concomitant variation of two characters or traits; (2) the interdependence of two variables; (3) the tendency for two variables to be related in a single-valued mathematical function. Correlation may be between two or more quantitative or non-quantitative variables. It may be linear (rectilinear, p. 73) or non-linear (curvilinear, p. 87).

The following **types of correlated variability** or relationship are considered in this chapter:

A. Correlated variability between two sets of variables.

- I. Both variables quantitative.
  - 1. Linear correspondence.
    - *a.* Computation of r with ungrouped data.
      - α. By deviations from true mean.
      - β. By deviations from $M_x = 0$ and $M_y = 0$.
      - γ. By rank difference method.
    - *b.* Computation of r from grouped data.
    - *c.* Regression lines.
  - 2. Non-linear correspondence: correlation ratio (η).
  - 3. Spurious correlation.
  - 4. Special coefficients.
    - *a.* Alienation.
    - *b.* Determination.
    - *c.* Reliability.
    - *d.* Attenuation.
    - *e.* The coefficient of similarity.
    - *f.* The correlation between a variable and the deviation of a dependent variable from its probable value.
  - 5. Analysis of variance in correlation.

II. One variable quantitative; the other in two categories (biserial correlation coefficient).

III. Both variables are non-quantitative.

1. Both sets of variables occur in several classes (coefficient of mean square contingency).
2. One set of variables with several classes (biserial eta).
3. Each set of variables with two classes (tetrachoric correlation).

B. Interdependence between three or more sets of variables.

I. Coefficient of partial correlation.

1. Computation of *r*.
2. Partial sigma.
3. Regression equations.

II. Multiple correlation.

III. Coefficient of part correlation.

IV. Tetrad difference.

A. CORRELATED VARIABILITY BETWEEN TWO SETS OF VARIABLES

I. Both Variables Quantitative

As just stated, the correspondence between two quantitative series of correlated data may be linear or non-linear or curvilinear. The following scatter diagrams illustrate the difference between these types (Figs. 19, 20). For a refined test of linearity see page 88.

Age in Years																
16													1	1	5	
15										1	1	1	3	5	1	2
14										2	5	4	5			
13									1	6	3	2	2			
12								3	4	10	6	1				
11							3	5	4	5		1				
10						5	2	7	4	1						
9				1	3	3	6	4								
8				1	7	8	6									
7			2	7	12	2	1									
6	1	2	3	3	2	1										
5		2	2	1												
	970	1020	1070	1120	1170	1220	1270	1320	1370	1420	1470	1520	1570	1620	1670	1720

Stature in Millimeters

FIG. 19. CORRELATION OF STATURE AND AGE FOR BOYS 5 TO 16 YEARS OF AGE. BROOKLYN ORPHAN ASYLUM DATA, UNITED STATES AND NORTH EUROPEAN RACIAL GROUP. 1933. (RECTILINEAR TYPE.)

Not only is there shown a close relationship between age and stature for these data, but the functions are markedly linear in the tendency shown by the data to form a straight band across the table.

NOTE: For both tables, scale values indicated at each class are the midpoints of the class intervals.

1. *Linear Correspondence*

The **coefficient of correlation,** *r*, also called the **Pearson product moment coefficient,*** is an index of the degree of linear relationship between two sets of varying values. The coefficient *r* may have any value from -1 to $+1$. In positive correlation, relationship is such that large deviations from the mean of one variable tend to be associated with large deviations of the other, small deviations with small deviations, etc. In negative correlation, relationship is reciprocal, and large deviations in one sense of one variable tend to be associated with corresponding deviations in the opposite sense of the

* After Karl Pearson, who developed the product-moment formula.

other. Zero correlation, $r = 0$, denotes the absence of any relationship between the variability of the series of variables.

Age in Years											
19	1										
18											
17	1										
16	1		2			1					
15		2	1	3							
14	3	1	4	1	1						
13	1	2	2		1						
12		3	4	1	1						
11	1	2	1	4							
10	1		3		1						
9		2	1	1	2						
8	1	1	1	2		2	1	1			
7		1	1		3		3	1			
6					1	1		1	3		
5						2		1	1		1
4							2	1			
3											1
	48.5	50.5	52.5	54.5	56.5	58.5	60.5	62.5	64.5	66.5	68.5

Thigh Girth/Leg Length-(Per Cent)

FIG. 20. CORRELATION OF AGE AND THIGH GIRTH/LEG LENGTH FOR BOYS 3 TO 19 YEARS OF AGE. BROOKLYN ORPHAN ASYLUM DATA. NORTH EUROPEAN RACIAL GROUP. 1933. (CURVILINEAR TYPE.)

Although relationship may be fairly high, the data show a tendency to form a curved band across the table. The table is definitely but not highly curvilinear. It is sometimes difficult to determine by inspection whether data are linear or curvilinear. See page 88 for more precise method.

a. Computation of r with ungrouped data. α. *By deviations from true mean.*

The fundamental formula is

$$r_{xy} = \frac{\Sigma xy}{N \sigma_x \sigma_y}$$

Example. The relation of body weight in kilograms (X) and transverse chest diameter in millimeters (Y) for 10 individuals will be used as an illustration. Desired: the correlation between these two dimensions.

TABLE 4.—RELATION BETWEEN WEIGHT AND TRANSVERSE CHEST DIAMETER IN 10 INDIVIDUALS, TOGETHER WITH THE COMPUTATION OF THE ELEMENTS OF THE COEFFICIENT OF CORRELATION

Let X = weight in kilograms.
Y = chest transverse diameter in millimeters.
M_x = arithmetic mean of the weight distribution.
M_y = arithmetic mean of chest diameter distribution.
σ_x = standard deviation of weight distribution.
σ_y = standard deviation of chest diameter distribution.
x = deviation of an individual's weight from the mean weight.
y = deviation of an individual's chest diameter from the mean chest diameter.
Σ = sum of, summation.
N = number of individuals in distribution.

Individual	X	Y	x	y	x^2	y^2	$\frac{x}{\sigma_x}$	$\frac{y}{\sigma_y}$	$\frac{xy}{\sigma_x\sigma_y}$	xy
1	45	226	−10.40	−18.10	108.16	327.61	−1.29	−0.82	1.05	188.24
2	44	221	−11.40	−23.10	129.96	533.61	−1.41	−1.04	1.47	263.34
3	66	281	10.60	36.90	112.36	1361.61	1.31	1.66	2.17	391.14
4	64	256	8.60	11.90	73.96	141.61	1.06	0.54	0.57	102.34
5	64	268	8.60	23.90	73.96	571.21	1.06	1.08	1.14	205.54
6	60	258	4.60	13.90	21.16	193.21	0.57	0.63	0.36	63.94
7	55	253	− 0.40	8.90	0.16	79.21	−0.05	0.40	−0.02	−3.56
8	47	221	− 8.40	−23.10	70.56	533.61	−1.04	−1.04	1.08	194.04
9	49	210	− 6.40	−34.10	40.96	1162.81	−0.79	−1.54	1.22	218.24
10	60	247	4.60	2.90	21.16	8.41	0.57	0.13	0.07	13.34
Total (Σ)	554	2441			652.40	4912.90			9.11	1636.60

$$r_{xy} = \frac{1636.60}{10 \times 8.08 \times 22.17} = 0.914.$$

The problem above may be worked without the use of the three columns, $\frac{x}{\sigma_x}$, $\frac{y}{\sigma_y}$ and $\frac{xy}{\sigma_x\sigma_y}$. To compute r the data in the right-hand column, combined with the σ's of the two arrays, will suffice, and $r = \frac{\Sigma xy}{N\sigma_x\sigma_y}$, which is the original form of the product-moment coefficient. Substituting in this formula,

$$r_{xy} = \frac{1636.60}{(10)(8.08)(22.17)} = +0.914.$$

To test the significance of an observed r, calculate

$$t = \frac{r}{\sqrt{1 - r^2}} \cdot \sqrt{N - 2}$$

where N is the number of paired observations on which r is based. Then enter Table XIII with $N - 2$ as the argument "degrees of freedom for smaller mean square." r is significant if the calculated t is equal to or greater than the value of t appearing in fine print, since an r of this magnitude would not occur more than five times in a hundred trials where the successive samples were taken from a population having a correlation of zero.

Example. In the problem on pages 80 and 81 where r_{xy} is 0.776

$$t = \frac{0.776}{\sqrt{1 - 0.776^2}} \sqrt{50 - 2} = 3.06$$

From Table XIII at 50 degrees of freedom the t value in bold faced type is 2.678. Since the value of t found is greater than this we conclude that r is significant, since an r of that size would not be found in an uncorrelated population as often as once in 100 trials.

To test whether the difference between two correlation coefficients r_1 and r_2, is significant calculate:

$$z'_1 = \tfrac{1}{2}\left\{\log_e(1 + r_1) - \log_e(1 - r_1)\right\}$$
$$z'_2 = \tfrac{1}{2}\left\{\log_e(1 + r_2) - \log_e(1 - r_2)\right\}$$

The standard error of $(z'_1 - z'_2)$ is:

$$\text{S.E.}(z'_1 - z'_2) = \mp \frac{1}{\sqrt{N_1 + N_2 - 6}}$$

where N_1 and N_2 are the number of paired observations on which r_1 and r_2 are based. If $(z'_1 - z'_2)$ is at least two times greater than its standard error the difference is probably significant.

A widely used formula for determining the standard error of an observed r is:

$$\text{S.E.}_{\cdot r} = \frac{1 - r^2}{\sqrt{N}}$$

This formula is quite accurate when N is large and r is small. For standard errors of various values of r consult Table XIV.

β. By deviations from means taken at $M_x = 0$ and $M_y = 0$, one proceeds as follows (same example):

Individual	X	Y	X^2	Y^2	XY
1.	45	226	2025	51076	10170
2.	44	221	1936	48841	9724
3.	66	281	4356	78961	18546
4.	64	256	4096	65536	16384
5.	64	268	4096	71824	17152
6.	60	258	3600	66564	15480
7.	55	253	3025	64009	13915
8.	47	221	2209	48841	10387
9.	49	210	2401	44100	10290
10.	60	247	3600	61009	14820
Totals, Σ =	554	2441	31344	600761	136868

$$M_x = \frac{\Sigma X}{N} = \frac{554}{10} = 55.4; \quad M_y = \frac{\Sigma Y}{N} = \frac{2441}{10} = 244.1.$$

$$\sigma_x = \sqrt{\frac{\Sigma X^2}{N} - M_x^2} = \sqrt{\frac{31344}{10} - 55.4^2} = 8.08.$$

$$\sigma_y = \sqrt{\frac{\Sigma Y^2}{N} - M_y^2} = \sqrt{\frac{600761}{10} - 244.1^2} = 22.17.$$

$$\frac{\Sigma xy}{N} = \frac{\Sigma XY}{N} - M_xM_y = \frac{136868}{10} - (55.4)\,(244.1) = 163.66 \text{ (numerator)}.$$

$$r_{xy} = \frac{\dfrac{\Sigma XY}{N} - M_xM_y}{\sigma_x\,\sigma_y} = \frac{163.66}{(8.08)\,(22.17)} = 0.914.$$

Still another formula which does not involve computing the means, thus eliminating fractions, is given by

$$r_{xy} = \frac{N\Sigma XY - \Sigma X\Sigma Y}{\sqrt{[N\Sigma X^2 - (\Sigma X)^2]\,[N\Sigma Y^2 - (\Sigma Y)^2]}}.$$

Substituting values of sums in the above problem gives

$$r_{xy} = \frac{10(136868) - (554)(2441)}{\sqrt{[10(31344) - (554)^2][10(600761) - (2441)^2]}} = 0.914.$$

A great number of variations of the original product-moment formula for r have been worked out for use with data of various forms (Symonds, 1926). One of the more commonly used formulæ is illustrated below.

γ. *The rank-difference method* for computing r is useful for short series or for series which may be ranked in order of merit or magnitude. It is less accurate than the product-moment method because it takes account only of relative position of the items in a series rather than the magnitude of the deviations from the mean. Use of this method is recommended when:

(1) One or both variables are expressed in ranks.

(2) A short method is desired for determining the presence of relationship rather than its extent.

(3) The nature and amount of the data do not warrant the use of more laborious methods giving accurate results.

Spearman's formula is as follows:

$$\rho = 1 - \frac{6\Sigma D^2}{N(N^2 - 1)}$$

where D refers to the differences in ranks. The method of assigning ranks to scores or measures must take account of possible ties in rank. Two methods are in use:

(1) The mid-rank method. Individual values are given equal rank in case of ties, and the rank assigned is the mid-rank of the set of equal values.

(2) The bracket-rank method. Individual values are given equal rank in case of ties, and the rank assigned is the value which would have been assigned to the first of the tied values, had there been no tie.

In both methods, the individual after the ties takes the rank which would have been assigned if there had been no ties.

Example.	Values	Rank Mid-rank method	Rank Bracket-rank method
	29	1	1
	30	2	2
	31	3.5	3
	31	3.5	3
	38	6	5
	38	6	5
	38	6	5
	42	8	8
	50	9	9

The following set of measures used on page 75 may serve to illustrate the method of computing ρ (rho):

		Ranks			
X	Y	x	y	$x - y$	$(x - y)^2$
44	221	1	2.5	−1.5	2.25
45	226	2	4	−2.0	4.00
47	221	3	2.5	0.5	0.25
49	210	4	1	3.0	9.00
55	253	5	6	−1.0	1.00
60	247	6.5	5	1.5	2.25
60	258	6.5	8	−1.5	2.25
64	256	8.5	7	1.5	2.25
64	268	8.5	9	−0.5	0.25
66	281	10	10	0	0
					$\Sigma D^2 = 23.50$

$$\rho = 1 - \frac{6(23.50)}{10(100 - 1)} = +0.86.$$

Assuming normally distributed variates, r is slightly larger than ρ. The correction formula given by Pearson is $r = 2 \sin \frac{\pi}{6} \rho$. Table XVII, page 189, presents values of r for computed values of ρ. For the above problem, with $\rho = +0.86$, $r = +0.87$.

The probable error of ρ is given by Pearson's formula

$$PE_\rho = \left[\frac{0.6745(1 - \rho^2)}{\sqrt{N}}\right] [1 + .086\rho^2 + 0.013\rho^4 + 0.002\rho^6].$$

For the problem above

$$PE_\rho = \left[\frac{0.6745(1 - 0.86^2)}{\sqrt{10}}\right][1 + .086(0.86^2) + 0.013(0.86^4) + 0.002(0.86^6)] = \pm 0.060.$$

If fine accuracy is not desired, the last term of the formula may be neglected and the formula becomes $PE_\rho = \frac{0.6745(1 - \rho^2)}{\sqrt{N}}$, which for our problem gives $PE_\rho = \pm 0.056$.

b. Computation of r with Grouped Data. *α. Standard Formula.* The formula for the calculation of the correlation coefficient r from grouped data and using assumed means is as follows:

$$r_{xy} = \frac{\Sigma fx'y' - \frac{(\Sigma fx')(\Sigma fy')}{N}}{\sqrt{\Sigma f(x')^2 - \frac{(\Sigma fx')^2}{N}}\sqrt{\Sigma f(y')^2 - \frac{\Sigma (fy')^2}{N}}}.$$

Example.

TABLE 5. — DISTRIBUTIONS OF THE MEASURES OF WEIGHT AND TROCHANTER BREADTH FOR 50 INDIVIDUALS; ILLUSTRATING METHOD OF WORKING OUT THE FORMULA FOR r_{xy}

Trochanter Breadth in Millimeters, x-variable; Weight in Kilograms, y-variable

Weight in Kilograms (y-variable)	295–299	300–304	305–309	310–314	315–319	320–324	325–329	330–334	335–339	340–344	345–349	350–354	355–359	360–364	365–369	370–374	375–379	380–384	f	y'	fy'	fy'^2	$\Sigma fx'y'$
70–71																		1	1	8	8	64	72
68–69																			0	7	0	0	0
66–67											1								1	6	6	36	12
64–65									1	1					1		1		4	5	20	100	75
62–63								1			1		2						4	4	16	64	36
60–61						1				1				1					3	3	9	27	9
58–59										1	1								2	2	4	8	6
56–57							2	1	1										4	1	4	4	−5
54–55					1				2	1	2								6	0	0	0	0
52–53									1										1	−1	−1	1	0
50–51			1	1		1	3								1				7	−2	−14	28	28
48–49			1		1	1	3		1										7	−3	−21	63	57
46–47			1			3													4	−4	−16	64	60
44–45		1		3															4	−5	−20	100	110
42–43	1																		1	−6	−6	36	48
40–41				1															1	−7	−7	49	35
																			50 N		−18 $\Sigma fy'$	644 $\Sigma fy'^2$	543 $\Sigma fx'y'$
f	1	1	3	5	2	6	8	2	6	4	5	0	2	1	2	0	1	1	50	N			
x'	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9					
fx'	−8	−7	−18	−25	−8	−18	−16	−2	0	4	10	0	8	5	12	0	8	9	−46	$\Sigma fx'$			
fx'^2	64	49	108	125	32	54	32	2	0	4	20	0	32	25	72	0	64	81	764	$\Sigma fx'^2$			

Let x' and y' = deviations in classes from the assumed mean of the x-variable and the y-variable, respectively.

fx', fy' = number of cases at a given class *times* the deviation of that class from the assumed mean.

fx'^2, fy'^2 = $(x')(fx')$ and $(y')(fy')$, respectively.

At the weight class (44–45) there are 4 cases, $f = 4$. This class is the fifth below the assumed mean, $y' = -5$. Then $fy' = 4(-5) = -20$; and $fy'^2 = (-5)(-20) = +100$.

$fx'y'$ = product-moments of frequencies about the assumed mean. The value of $fx'y'$ for each row is the sum of the products of the cell frequencies by their corresponding deviations. *For example*, in class 70–71, one case is plotted in the cell at x' deviation, $+9$, and at y' deviation, $+8$, hence $fx'y' = (1)\ (+9)\ (+8) = 72$. In class 62–63, *one* case is plotted at $x' = -1$ and $y' = +4$, *one* at $x' = +2$ and $y' = +4$, *two* at $x' = +4$ and $y' = +4$, hence $fx'y' = (1)(-1)(+4) + (1)(+2)(+4) + (2)(+4)(+4) = +36$.

Substituting in the formula:

$$r_{xy} = \frac{543 - \dfrac{(-46)(-18)}{50}}{\sqrt{764 - \dfrac{(-46)^2}{50}}\sqrt{644 - \dfrac{(-18)^2}{50}}} = 0.776.$$

β. Sum and Difference Formulæ. The summation method may be applied to correlation problems and is quite rapid for use with an adding and calculating machine. The formulas are

$$r_{xy} = \frac{\sigma^2_x + \sigma^2_y - \sigma^2_{x-y}}{2\sigma_x\sigma_y} \text{ (difference formula).}$$

$$r_{xy} = \frac{\sigma^2_{x+y} - \sigma^2_x - \sigma^2_y}{2\sigma_x\sigma_y} \text{ (sum formula).}$$

The moments σ^2_{x-y} and σ^2_{x+y} are obtained by summing the frequencies along diagonals from lower left to upper right and from lower right to upper left, respectively, when the lower limits of the variables are located at the lower left corner of the scatter diagram. The difference formula involves least work when correlation is positive and high; the sum formula is shorter when correlation is negative and high. The moments may be checked by working out both numerators. The sample problem illustrates the summation method using the difference formula:

Let x', y', $(x - y)'$ = cumulative sum of frequencies at each class.

x'', y'', $(x - y)''$ = cumulative sum of x', y', $(x - y)'$, respectively, at each class.

TABLE 6.—CORRELATION BETWEEN WEIGHT AND FOOT AREA TO ILLUSTRATE THE APPLICATION OF THE DIFFERENCE FORMULA

$f(x-y)$	$(x-y)'$	$(x-y)''$
1	55	264
1	54	209
4	53	155
11	49	102
25	38	53
11	13	15
2	2	2
55	264	800
N	$\Sigma(x-y)'$	$\Sigma(x-y)''$

Weight in kg									fy	y'	y''
65–69								1	1	55	330
60–64							1		1	54	275
55–59				1			1	3	5	53	221
50–54					1		2	1	4	48	168
45–49			1			4	2	1	8	44	120
40–44				5	4	4			13	36	76
35–39		1	1	5	3	1			11	23	40
30–34		3	3	1					7	12	17
25–29	1	4							5	5	5
Foot Area sq. cm.	145–159	160–174	175–189	190–204	205–219	220–234	235–249	250–264	55 N	330 $\Sigma y'$	1252 $\Sigma y''$
fx	1	8	5	12	8	9	6	6	55	N	
x'	1	9	14	26	34	43	49	55	231	$\Sigma x'$	
x''	1	10	24	50	84	127	176	231	703	$\Sigma x''$	

$$\sigma^2_x = \frac{2\,\Sigma x''}{N} - \frac{\Sigma x'}{N} - \left(\frac{\Sigma x'}{N}\right)^2.$$

$$\sigma^2_x = \frac{2(703)}{55} - \frac{231}{55} - \left(\frac{231}{55}\right)^2 = 3.724.$$

$$\sigma^2_y = \frac{2\,\Sigma y''}{N} - \frac{\Sigma y'}{N} - \left(\frac{\Sigma y'}{N}\right)^2.$$

$$\sigma^2_y = \frac{2(1252)}{55} - \frac{330}{55} - \left(\frac{330}{55}\right)^2 = 3.527$$

$$\sigma^2_{(x-y)} = \frac{2\,\Sigma(x-y)''}{N} - \frac{\Sigma(x-y)'}{N} - \left(\frac{\Sigma(x-y)'}{N}\right)^2;$$

$$\sigma^2_{(x-y)} = \frac{2(800)}{55} - \frac{264}{55} - \left(\frac{264}{55}\right)^2 = 1.251.$$

$$\sigma_x = \sqrt{3.724} = 1.930.^*$$

$$\sigma_y = \sqrt{3.527} = 1.878.^*$$

$$r_{xy} = \frac{3.724 + 3.527 - 1.251}{(2)\,(1.930)\,(1.878)} = +0.828.$$

c. Regression Lines. When a correlation table is formed as in Fig. 19, page 73, the surface that is covered with figures has a varied form, but is typically some form of ellipse. At one extreme the surface is a circle; at the other nearly a band whose long axis runs across the table at 45° (Fig. 19). That long axis is called a *regression* line.

Two regression lines can be drawn through every correlation surface: one plotted for the mean y value for every array of x; and one plotted for the mean x value for every array of y. These lines cross somewhere near the middle of the figure. The slopes of the lines are measured by the *regression coefficients*, of which there are two: $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$; $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$, where σ_x and σ_y are the respective standard deviations of the distributions of x and y, and r is the coefficient of correlation. The higher the correlation in any scatter the more alike the slope of the two regression lines. Two regression coefficients may both be less than 1. Their sum may be greater than 1. Their product equals r^2. Both have the same sign as r. Both can not be greater than 1 (Fig. 21).

The regression equation to predict Y from X is

$$Y = a + bX$$

where a is a constant, and b is the slope of the regression line or the change in Y per unit change in X. The constants a and b are obtained as follows:

$$a = M_y - bM_x$$

$$b = r\frac{\sigma_y}{\sigma_x},$$ (σ_y and σ_x should be expressed in original units).

* These standard deviations, as used in the formula, are in terms of the classes as units. The standard deviation of the distribution is this value multiplied by the size of the class range.

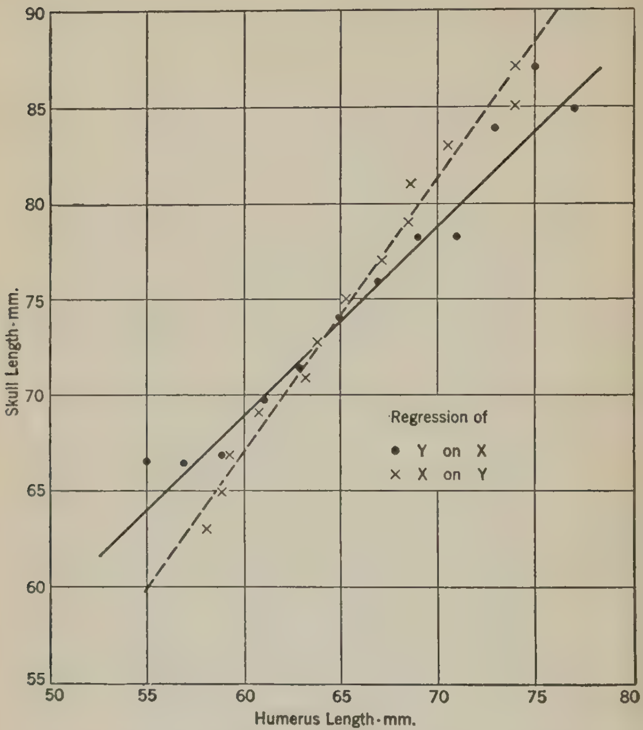

FIG. 21.—The two regression lines based on the correlation surface shown in Fig. 22 and Table 7. The black dots indicate the mean Y position of the X-arrays in that correlation surface taken at the center of the 2 mm. classes of x. The crosses indicate the mean X position of the points at the center of the 2 mm. classes of y. The solid line passes as closely as possible through the dots. It is the line of regression of Y on X. The broken line passing through the crosses is that of regression of X on Y.

The equation to predict X from Y is

$$X = a + bY$$

where

$$a = M_x - bM_y$$

$$b = \frac{\sigma_x}{\sigma_y} r \text{ (}\sigma_x \text{ and } \sigma_y \text{ should be expressed in original units).}$$

Example. The following data obtained from Table 7 illustrate the method of predicting Y from X.

Humerus length $= X$ $\quad M_x = 64.16$ mm. $\quad M_y = 73.01$ mm.

Skull length $= Y$ $\quad \sigma_x = 3.77$ mm. $\quad \sigma_y = 4.46$ mm.

$$r_{xy} = 0.834$$

$$b = \frac{4.46}{3.77}(0.834) = 0.987.$$

$$a = 73.01 - (0.987)(64.16) = 9.68.$$

$$Y = 9.68 + 0.987X.$$

TABLE 7.—CORRELATION BETWEEN HUMERUS LENGTH AND SKULL LENGTH IN RABBITS

Skull length	Humerus length													Total
	54	56	58	60	62	64	66	68	70	72	74	76	
62	....	2	3	....	....	....	....	....	...	...	...	...	5
64	1	5	5	5	....	....	....	....	...	...	...	...	16
66	4	4	7	3	7	....	....	....	...	...	...	...	25
68	....	5	8	13	14	2	....	....	...	...	...	...	42
70	....	....	....	14	23	14	3	....	...	...	...	...	54
72	....	....	....	5	22	11	8	....	1	...	...	...	47
74	....	....	1	2	9	23	18	3	2	...	...	...	58
76	....	....	....	....	3	12	21	11	5	...	...	...	52
78	....	....	....	....	....	2	6	12	4	...	...	...	24
80	....	....	....	....	....	....	4	4	2	...	...	...	10
82	....	....	....	....	....	....	....	1	2	1	...	...	4
84	....	....	....	....	....	....	....	....	...	1	...	1	2
86	....	....	....	....	....	....	....	....	...	...	1	...	1
Total	5	16	24	42	78	64	60	31	16	2	1	1	340

Mean humerus, 64.16 ± 0.13. *S.D.* humerus, 3.77 ± 0.09.
Mean skull, 73.01 ± 0.16. *S.D.* skull, 4.46 ± 0.11.
r, 0.834 ± 0.011.

The graph of this equation is shown in Fig. 22, page 86, where it is the solid line. The dots on the chart are the actual observations. The meaning of the lines indicated by dashes will be explained in the discussion of the standard error of estimate.

The standard error of estimate is a measure of how closely the predicted values obtained by a regression equation agree with the actual ones. It is really a standard deviation, but

in this case the deviations are measured from the regression line and not the mean.

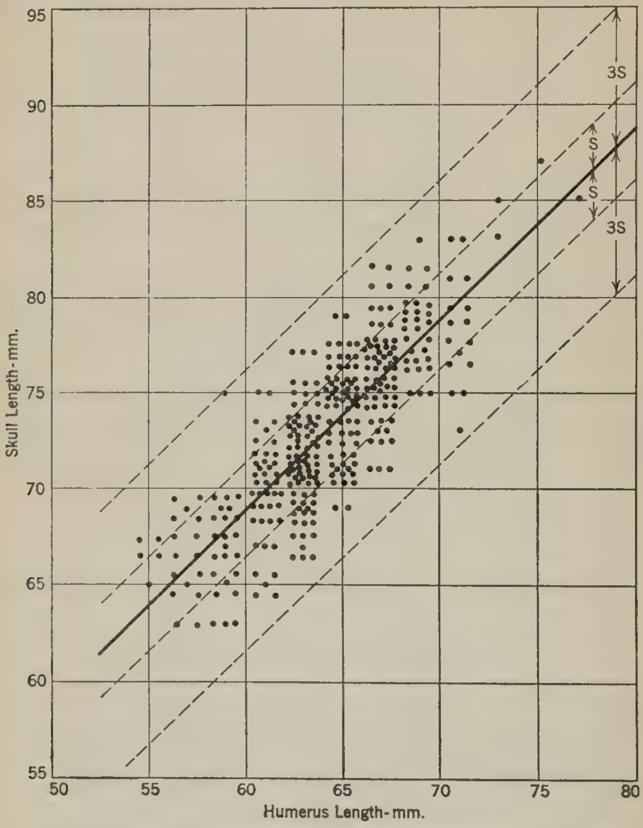

FIG. 22.—Graphic representation of (a) the equation, $Y = 9.68 + 0.987X$, being that of the regression line of the correlation surface between skull length and humerus length shown in Table 7 (full line) and (b) the standard error of the regression line (inner broken line) and (c) $3 \times$ standard error (outer broken line). See text, page 85.

The formula for computing the standard error of estimate of Y, when predicting Y from X, is

$$SE_{yx} = \sigma_y \sqrt{1 - r^2_{xy}}.$$

In case X is being predicted from Y, the formula becomes

$$SE_{yx} = \sigma_x \sqrt{1 - r^2_{xy}}.$$

Both σ_x and σ_y should be in original units.

In the present example,

$$SE_{yx} = 4.46 \sqrt{1 - 0.834^2} = 2.46 \text{ mm.}$$

The standard error of estimate when plotted and centered about the regression line gives the limits within which a certain proportion of the actual observations will fall. Since the standard error of estimate is really a standard deviation it may be interpreted in a similar way. Thus, if the observations are distributed normally about the regression line, 68 per cent of the cases will fall within the range indicated by $2SE$ when centered about the regression line. A range of $6SE$ will include about 99.7 per cent of the observation. (These proportions may be obtained from Table IV.)

Thus in Fig. 22 about 68 per cent of the observations will fall within the limits indicated by (2×2.46) millimeters, and about 99.7 per cent of them will fall within the limits indicated by (6×2.46) millimeters.

2. *Non-linear Correspondence*

Correlation Ratio. Eta (η) is a measure of correlation analogous to the correlation coefficient, but is used when the variables show curvilinear relationship. There are always two η's, the η_{yx} and η_{xy}. When the regression is strictly linear, $r_{xy} = \eta_{yx} = \eta_{xy}$; otherwise both η_{yx} and η_{xy} are greater than r_{xy}. When giving the results obtained by the correlation ratio analysis it is very important to indicate clearly whether the results pertain to η_{yx} or η_{xy}, since these are not the same as in linear correlation, where $r_{xy} = r_{yx}$.

η_{yx} is the ratio of the weighted standard deviation of the Y means to the standard deviation of all the Y values. A mean value of Y is calculated for each array of X; this is called M_{ya}. Then the standard deviations for these various means are calculated in the usual manner. It is important to remember

when calculating the standard deviations of the array means that each mean is given a weight equal to the frequency on which it is based. η_{xy} is interpreted in a way similar to that of η_{yx}.

The formulæ for computing the standard deviations of the array means are:

$$\sigma_{my} = \sqrt{\frac{\Sigma f_x(M_{ya} - M_y)^2}{N}}; \quad \sigma_{mx} = \sqrt{\frac{\Sigma f_y(M_{xa} - M_x)^2}{N}}.$$

where σ_{my} = the weighted standard deviation of the Y means for the different arrays of x.
fx = the frequency on which each Y mean is based.
M_y = the mean of all the Y observations.
N = the total number of observations.

The formula for obtaining σ_{mx} has a meaning similar to that of σ_{my}.

The formulæ for computing η are

$$\eta_{yx} = \frac{\sigma_{my}}{\sigma_y}; \quad \eta_{xy} = \frac{\sigma_{mx}}{\sigma_x}.$$

No sign is given to η, since the relationship may be positive for one part of a distribution and negative for another. The value of η, like r, can not exceed the value 1.

The value of η is not independent of the number of arrays into which the observations have been grouped. If there are as many arrays as there are observations, η necessarily becomes 1, and is therefore meaningless in such a case. If a population in which the association is zero is grouped into K arrays, η^2 tends to approach the value $\frac{K - 1}{N - 1}$ and not zero.

To test whether an observed η is significant make use of formula (a). To test whether for a given sample r and η are significantly different use (b).

$$(a)\ F = \frac{\eta^2(N - K)}{(1 - \eta^2)(K - 1)} \qquad (b)\ F = \frac{\eta^2 - r^2}{1 - \eta^2} \cdot \frac{N - K}{K - 2}$$

Enter Table XIII with $K - 1$ (or $K - 2$) equal to degrees of freedom for the greater mean square (p. 38).

Example.

TABLE 8.—CORRELATIONS BETWEEN AGE AND THIGH GIRTH/LEG LENGTH RATIO FOR BOYS

$\frac{\text{THIGH GIRTH}}{\text{LEG LENGTH}} \times 100$

AGE IN YEARS	48	50	52	54	56	58	60	62	64	66	68	fy	Array mean M_{xa}	$M_{xa} - M_x$	$(M_{xa} - M_x)^2$	$fy(M_{xa} - M_x)^2$
19	1											1	48.50	−6.36	40.4496	40.4496
18												0	....	....	....	....
17	1											1	48.50	−6.36	40.4496	40.4496
16	1		2			1						4	53.00	−1.86	3.4596	13.8384
15		2	1	3								6	52.83	−2.03	4.1209	24.7254
14	3	1	4	1	1							10	51.70	−3.16	9.9856	99.8560
13	1	2	2		1							6	51.83	−3.03	9.1809	55.0854
12		3	4	1	1							9	52.50	−2.36	5.5696	50.1264
11	1	2	1	4								8	52.50	−2.36	5.5696	44.5568
10	1		3		1							5	52.50	−2.36	5.5696	27.8480
9		2	1	1	2							6	53.50	−1.36	1.8496	11.0976
8	1	1	1	2		2	1	1				9	55.61	.75	.5625	5.0625
7		1	1		3		3	1				9	57.39	2.53	6.4009	57.6081
6					1	1		1	3			6	61.83	6.97	48.5809	291.4854
5						2		1	1		1	5	62.50	7.64	58.3696	291.8480
4							2	1				3	61.17	6.31	39.8161	119.4483
3											1	1	68.50	13.64	186.0496	186.0496
$f_x \rightarrow$	10	14	20	12	10	6	6	5	4		2	89 N		$M_x = 54.86$		$\Sigma = 1359.5351$
Array mean M_{ya}	13.60	11.50	12.10	11.67	9.40	8.00	6.17	6.00	5.75	....	4.00					
$M_{ya} - M_y$	3.27	1.17	1.77	1.34	− .93	−2.33	−4.16	−4.33	−4.58	....	−6.33					
$(M_{ya} - M_y)^2$	10.6929	1.3689	3.1329	1.7956	.8649	5.4289	17.3056	18.7489	20.9764	....	40.0689					
$f_x(M_{ya} - M_y)^2$	106.9290	19.1646	62.6580	21.5472	8.6490	32.5734	103.8336	93.7445	83.9056	....	80.1378					

$M_y = 10.33$ $\Sigma = 613.1427$

$$\sigma_{mx} = \sqrt{\frac{\Sigma fy(M_{xa} - M_x)^2}{N}} = \sqrt{\frac{1359.5351}{89}} = 3.9084$$

$$\eta_{xy} = \frac{\sigma_{mx}}{\sigma_x} = \frac{3.9084}{4.8782} = 0.8012. \qquad \eta_{yx} = \frac{\sigma_{my}}{\sigma_y} = \frac{2.6247}{3.6316} = 0.7227:$$

$$\sigma_{my} = \sqrt{\frac{\Sigma fx(M_{ya} - M_y)^2}{N}} = \sqrt{\frac{613.1427}{89}} = 2.6247$$

For age: mean = 10.33; σ = 3.6316. For thigh girth × 100 ÷ leg length: mean = 54.86; σ = 4.8782.

3. *Spurious Correlation*

Spurious correlation between variables is correlation due partly or wholly to factors other than those on which the true value of the coefficient of correlation is based. Spurious correlation may arise from one or more of the following:

1. Presence of one or more common factors in both variables. E.g., correlation of physical growth with mental growth would give high positive value for r because of the common factor, *age*. To eliminate spurious value, use the partial r technique (p. 108).

2. Correlation of a single measure with a composite which includes it. E.g., the correlation of leg length with stature. The amount of spurious correlation present is given by $\frac{M_a}{M_{a+b}}$, where M_a = mean of the single measure, and M_{a+b} = mean of the composite.

3. Increased range of material. E.g., suppose that two variables have scales of 0 to 100. Given a series of measurements none of which exceeds magnitude 50 on either scale, and with the coefficient of correlation between the two variables = 0. If a second series of measurements is made, also showing zero correlation, but lying in the 50–100 range of the scales, combining the two distributions will give a surface for which the correlation coefficient may be well above zero. Mills (1924, p. 407) gives an extreme example where the addition of *one case* high up on the scales raised an obtained r from -0.034 to $+0.999$!

4. Correlation of indices. Pearson (1897) has shown that spurious correlation may arise when indices are correlated. The amount of spurious correlation present may be as high as 0.500. The denominator of each index becomes a common factor contributing the spurious correlation. E.g., a correlation of relative span (i.e., span/stature) with relative sitting height (i.e., sitting height/stature) will show considerable spurious correlation because of the common denominator, stature, in each index. A measure of the spurious correlation arising when indices are correlated is given by the formula:

$$r_0 = \frac{V_c^2}{\sqrt{V_a^2 + V_c^2}\sqrt{V_b^2 + V_c^2}} \quad \text{(spurious correlation in indices)}$$

where r_0 = the amount of spurious correlation between the indices $\frac{a}{c}$ and $\frac{b}{c}$.

V_a = the coefficient of variation, $\frac{\sigma_a}{M_a} \times 100$, of the a factor, i.e., the numerator of the first index.

V_b = the coefficient of variation, $\frac{\sigma_b}{M_b} \times 100$, of the b factor, i.e., the numerator of the second index.

V_c = the coefficient of variation, $\frac{\sigma_c}{M_c} \times 100$, of the c factor, i.e., the denominator of both indices.

When using the formula it is assumed that the absolute values, a, b and c are uncorrelated.

4. *Special Coefficients*

a. The coefficient of alienation measures lack of correlation between two variables. Kelley (1919, p. 173) has assigned the symbol k to denote this coefficient, and its formula is

$$k = \sqrt{1 - r^2} \text{ (coefficient of alienation)}$$

The coefficient k is also called residual correlation by Tryon (1929), k^2 being "the degree of determination of Y from residual factors other than X." The coefficient k may be used as a measure of the predictive value of an r. The estimate improves as r increases. Table 9 gives values of k for values of r from 0.10 to 1.00. E.g., to reduce the error of estimate by one-half, $k = 0.50$, r must be 0.866. Again, r must be 0.98 before the error of estimate is reduced to one-fifth of a chance prediction ($k = 0.199$). The general relation between r and k is $r^2 + k^2 = 1$.

When $k = 1$ and $r = 0$, prediction of a score by means of the regression equation becomes merely a "chance" prediction.

Holzinger (1928, p. 166) gives a further application in prediction with his formula

$$I_p = 100(1 - \sqrt{1 - r^2}) \text{ (improvement over change in prediction by a single score)}$$

E.g., for $r = 0.50$, $I_p = 13.4$, which means that the regression forecast with a single score when $r = 0.50$ is only 13.4 per cent better than a random guess.

TABLE 9.—GIVING VALUES OF k FOR VALUES OF r FROM 0.0 TO 1.00

Coefficient of correlation r	Coefficient of alienation k	Coefficient of correlation r	Coefficient of alienation k
0.00	1.000	.....	.....
.10	0.995	0.80	0.600
.20	.980	.85	.527
.30	.954	.866	.500
.40	.917	.90	.436
.50	.866	.95	.312
.60	.800	.98	.199
.70	.714	.99	.141
.7071	.7071	1.00	.000

b. The coefficient of determination is the square of the coefficient of correlation, i.e., r^2. It gives a measure of the per cent of variation in the dependent variable associated with the independent variable. It may be shown (Ezekiel, 1930, p. 120) that when half the variation in Y is directly associated with X, $r = \sqrt{\frac{1}{2}} = 0.707$. By the table above, $k = 0.707$ also. But $r^2 = 0.707^2 = 0.5$, which indicates that 50 per cent of the variation in Y is *determined* by X. Then, also, k^2 (called the coefficient of non-determination) is $0.707^2 = 0.50$, the per cent of variation in Y *not determined* by X. For further uses of coefficients of determination, see Ezekiel (1930).

c. The Coefficient of Reliability. A reliable test or measuring instrument should give an identical set of scores when applied to the same individuals a second time (assuming that the individuals have not changed). Correlating such a first and second set of scores would give a coefficient of +1.00, showing perfect reliability. Since few instruments are perfectly accurate, the correlation coefficient computed for two sets of scores or two forms of a test of the same individuals is a measure of the degree of accuracy and is called a "coefficient of reliability."

The formula that gives this coefficient of reliability is:

$$r(z_1 + z'_1)\,(z_I + z'_I) = \frac{2r_{1I}}{1 + r_{1I}}$$

where $z_1 = \dfrac{x_1}{\sigma_1}$ and $z'_1 = \dfrac{x'_1}{\sigma'_1}$ are the standard scores (or variates) obtained by two similar tests (observations), a and b, and $z_{\mathrm{I}} = \dfrac{x_{\mathrm{I}}}{\sigma_{\mathrm{I}}}$, $z'_{\mathrm{I}} = \dfrac{x'_{\mathrm{I}}}{\sigma'_{\mathrm{I}}}$ are the standard scores (or variates) on another form of the same test (observational technique) α and β. Also, $r_{1\mathrm{I}}$ is the correlation between the first scores by each method of testing.

Repeating the test several times or lengthening a test tends to increase reliability. The Spearman (1913)-Brown (1911) prophecy-formula predicts the size of the new reliability coefficient when a test is lengthened:

$$r = \frac{Nr_{x_1x_2}}{1 + (N - 1)r_{x_1x_2}} \quad \text{(Spearman-Brown prophecy-formula)}$$

where r = the predicted reliability coefficient.
$r_{x_1x_2}$ = the obtained reliability coefficient.
N = the number of times length of test is increased (i.e., doubling the length of test: $N = 2$, etc.)

Suppose that an intelligence test containing 85 items has a reliability coefficient, $r_{x_1x_2} = 0.882$. Desired: the reliability coefficient if the test is increased to a length of 150 items:

$$N = \frac{150}{85} = 1.765 \qquad r = \frac{(1.765)\,(0.882)}{1 + (1.765 - 1)\,(0.882)} = 0.930.$$

The formula may be used also to determine how much a test should be lengthened (or how many times repeated) to produce a desired reliability coefficient.

$$N = \frac{r - rr_{x_1x_2}}{r_{x_1x_2} - rr_{x_1x_2}} \quad \text{(formula to predict length of test for desired reliability coefficient).}$$

Desired for the above test a reliability coefficient of 0.950; then

$$N = \frac{0.950 - (0.950)\,(0.882)}{0.882 - (0.950)\,(0.882)} = 2.542$$

and the test should contain at least 217 items (since $85 \times 2.542 = 216+$).

d. Correction for Attenuation. If, to check the error of individual measurements, the observations or measurements of the x and the y variables be repeated, giving x_1 and x_2 values, and also y_1 and y_2 values, the correlation coefficient, r, may be corrected for errors of measurement in the variables by using the formula of Spearman (1907) for correction for attenuation.

$$r'_{xy} = \sqrt{\frac{(r_{x_1y_2})\,(r_{x_2y_1})}{(r_{x_1x_2})\,(r_{y_1y_2})}}$$

where r'_{xy} = the corrected coefficient of correlation.

$r_{x_1y_2}$ = the correlation between a first set of measurements in the x variable and a second set of measures in the y variable.

$r_{x_2y_1}$ = the correlation between a second set of measures in x and a first set of measures in y.

$r_{x_1x_2}$ = the correlation between a first and second set of measures in x, i.e., the reliability coefficient of the x measure.

$r_{y_1y_2}$ = the correlation between a first and second set of measures in y, i.e., the reliability coefficient of the y measure.

Example. Let $r_{x_1x_2} = 0.842$; $r_{y_1y_2} = 0.785$; $r_{x_1y_2} = 0.684$; $r_{x_2y_1} = 0.726$.

Then
$$r_{xy} = \sqrt{\frac{(0.684)\,(0.726)}{(0.842)\,(0.785)}} = 0.867.$$

Given only one correlation between the two variables, an approximate correction may be made, if the reliability coefficients are known, by the formula

$$r_{xy} = \frac{r_{x_1y_1}}{\sqrt{(r_{x_1x_2})\,(r_{y_1y_2})}} \text{ (Spearman, 1904).}$$

where $r_{x_1y_1}$ = the obtained correlation between the two variables, and the other terms in the formula have the same definitions as above. If the obtained correlation $r_{x_1y_1} = 0.700$ then

$$r_{xy} = \frac{0.700}{\sqrt{(0.842)\,(0.785)}} = 0.861$$

Errors of attenuation lower r except when the errors are correlated, i.e., chance errors of measurement reduce r, but constant errors leave r unchanged.

e. The coefficient of similarity (Sm), also called the **first moment correlation coefficient,** may be used to measure relationship when data are not normally distributed and certain extreme items give exaggerated values for standard deviations and product moments. The coefficient is based on average deviations. Davies (1930) claims it particularly useful in time series problems. Sm coincides with r at the -1 and $+1$ limits, but is generally less than r for other values of r. It does not give rise to a regression line for prediction purposes, although an approximation may be obtained by graphing.

f. The Correlation between a Variable and the Deviation of a Dependent Variable from Its Probable Value (J. A. Harris, 1909). Given two variables, X and Y, where Y (the dependent variable) is a part of X, it is useful to know not only whether Y increases as X increases, but also whether Y bears the same proportion to X as X increases from low to high values. To determine this it is only necessary to have the constants that are usually computed when finding the correlation between X and Y and to substitute them in the formula:

$$r_d = \frac{r_{xy} - V_x/V_y}{\sqrt{(1 - r^2_{xy}) + (r_{xy} - V_x/V_y)^2}}:$$

$$r_{xy} = \text{the correlation between } X \text{ and } Y.$$

$$V_x = \frac{\text{standard deviation of } X}{\text{mean of } X} \times 100.$$

$$V_y = \frac{\text{standard deviation of } Y}{\text{mean of } Y} \times 100.$$

If r_d is greater than zero it means that Y becomes relatively greater as compared with X, as X increases from low to high values; and $r_{xy} > V_x/V_y$.

If r_d is less than zero it means that Y becomes relatively smaller as compared with X as X increases from low to high values; and $r_{xy} < V_x/V_y$.

If r_d is zero, it means that Y maintains the same proportion to X as X increases from low to high values.

For examples, Harris counted the number of ovules per pod in a large series of black locust trees; and he distinguished between these ovules which had developed into seed and those which had not. He then sought the relationship between the relative ability of pods with a large number of ovules to mature their seeds and those with a smaller number. The data are from a sample of 1427 pods of the black locust collected from 12 trees.

$$X = \text{ovules.} \qquad Y = \text{seeds.}$$

$$M_x = 12.1794. \qquad M_y = 7.6874.$$

$$\sigma_x = 2.2763. \qquad \sigma_y = 3.4938.$$

$$r_{xy} = 0.693 \pm 0.009.$$

From the above data:

$$V_x = \frac{\sigma_x}{M_x} = \frac{2.2763}{12.1794} = 18.690\%.$$

$$V_y = \frac{\sigma_y}{M_y} = \frac{3.4938}{7.6874} = 45.448\%.$$

$$r_d = \frac{r_{xy} - V_x/V_y}{\sqrt{1 - r^2_{xy} + (r_{xy} - V_x/V_y)^2}}$$

$$= \frac{0.693 - 0.411}{\sqrt{1 - 693^2 + (0.693 - 0.411)^2}} = 0.364$$

where V indicates coefficient of variation and r_d is as on page 95. This is the same formula as given on the preceding page. Harris concludes that an r_d of 0.364 indicates that for this sample the pods with the larger number of ovules are relatively more capable of maturing their seeds than those with fewer.

5. *Analysis of Variance in Correlation*

In any correlation table like that on page 85 we can distinguish two variables: (1) the variance of the straight regres-

sion line (passing in this case from upper left to lower right); and (2) the variance within the array. (1) is measured (*a*) by the summation of the squared deviations of each of the y (regression) values from the y mean corresponding to each x class; or (*b*) r^2 being known, by the product of r^2 and the sum of the squared deviations of each of the y class entries from the mean of all the y values, all divided by the number of degrees of freedom (see p. 31). (2) is measured (*a*) by the summation of squared deviations of each of the y class entries from the corresponding Y (regression) values; or (*b*) r being known, it is the product of $(1 - r^2)$ by the summation of the squared deviations of each of the y class entries from the mean of all the y values, all divided by the number of degrees of freedom. Or, 1*b* having been calculated, 2*b* is equal to 1*b* multiplied by $\dfrac{1 - r^2}{r^2} = \dfrac{1}{r^2} - 1.$

II. One Variable Quantitative; The Other in Two Categories

The **biserial correlation coefficient** is an index of relationship between two variables, one of which is expressed in a quantitative distribution while the other is expressed in two categories (Pearson, 1909). The assumptions are that the categorical data are really continuous and normal in distribution, and that the relationship existing between the variables is linear. The formula for the coefficient is:

$$r_{bis} = \frac{(M_2 - M_1)pq}{\sigma_y z}$$

where M_1, M_2 = respective means of first and second categorical distributions.

p, q = respective proportions of cases in larger and smaller distributions.

z = ordinate of normal probability curve at limit of area $p - 0.5$ (obtained from Table IIIa).

σ_y = standard deviation of Y_1 and Y_2 combined.

The probable error is

$$PEr_{bis} = \frac{0.6745\left(\frac{\sqrt{pq}}{z} - r^2_{bis}\right)}{\sqrt{N}}.$$

Example.

TABLE 10.—CORRELATION BETWEEN DISTRIBUTIONS OF BODY BUILD (WEIGHT/STATURE²) FOR THE OFFSPRING OF PARENTS WHOSE BUILD IS DESCRIBED CATEGORICALLY AS VERY SLENDER (*VS*), SLENDER (*S*), MEDIUM (*M*), FLESHY (*F*), AND VERY FLESHY (*VF*).

The distribution of progeny of different matings according to index of build is as follows:*

Build index	$VS \times S$	$VS \times M$	$S \times S$	$S \times M$	$M \times M$	$M \times F$	$M \times VF$	$F \times F$	$F \times VF$	$VF \times VF$	Totals
22	1	1	1	....	....	....	....	....	....	....	3
24	1	....	4	....	2	....	2	....	....	....	9
26	4	2	14	6	4	1	1	1	1	....	34
28	1	3	12	24	12	11	3	10	....	....	76
30	2	6	9	57	48	31	9	8	6	2	178
32	1	9	7	74	67	86	17	10	15	5	291
34	....	2	....	60	62	67	24	29	13	5	262
36	....	3	....	36	66	62	12	30	9	3	221
38	1	....	....	19	35	33	16	18	16	2	140
40	....	....	....	12	18	23	7	13	9	2	84
42	....	2	....	7	6	10	4	17	14	1	61
44	....	....	....	5	1	5	7	8	6	3	35
46	....	....	....	2	2	5	3	5	5	4	26
48	....	....	....	2	2	1	2	2	3	....	12
50	....	....	....	2	1	1	2	2	1	3	12
52	....	....	....	....	....	2	1	3	2	....	8
54	....	....	....	....	....	1	....	....	....	....	1
56	....	....	....	....	1	....	....	....	....	....	1
58	....	....	....	....	....	1	....	....	....	....	1
Totals	11	28	47	306	327	340	110	156	100	30	1455

* Adapted from Davenport (1923, p. 34).

A biserial distribution may be formed by dividing the table arbitrarily between $S \times M$ and $M \times M$, so that no fleshy parents contribute progeny to the first section of the distribution, and no slender parents contribute to the second section.

Let Y_1 = index of build distribution of progeny from "non-fleshy" parents
Y_2 = index of build distribution of progeny from fleshy parents.
Y = total distribution.

Build index	Matings Y_1	Y_2	Y
22	3	....	3
24	5	4	9
26	26	8	34
28	40	36	76
30	74	104	178
32	91	200	291
34	62	200	262
36	39	182	221
38	20	120	140
40	12	72	84
42	9	52	61
44	5	30	35
46	2	24	26
48	2	10	12
50	2	10	12
52	...	8	8
54	...	1	1
56	...	1	1
58	...	1	1
Totals	392	1063	1455

$p = 0.73058.$

$q = 0.26942.$

$z = 0.33029.$

$\sigma_y = 4.98996.$

$M_1 = 32.53061.$

$M_2 = 35.55597.$

$$r_{bis} = \frac{(35.55597 - 32.53061)(0.73058)(0.26942)}{(4.98996)(0.33029)}.$$

$r_{bis} = 0.3613.$

$$PEr_{bis} = \frac{0.6745\left(\frac{\sqrt{(0.73058)(0.26942)}}{0.33029} - 0.3613^2\right)}{\sqrt{1455}}.$$

$PEr_{bis} = \pm 0.0214.$

If the grouping is broad in the quantitative distribution, e.g., less than 12 intervals, Sheppard's (1898) correction

$$_c\sigma_y = \sqrt{\sigma_y^2 - \frac{i}{12}}$$

should be applied to the σ_y used in the denominator. In the correction formula, σ_y = the obtained standard deviation, i = the size of the class interval, and $_c\sigma_y$ = the corrected standard deviation.

A correction to the coefficient r_{bis} for small populations, e.g., less than 100, may be made with Soper's (1914) formula

$$_cr_{bis} = r_{bis}\left\{1 + \frac{1}{N}\left[\frac{1}{4} + \frac{pq}{2z^2} - \left(1 - \frac{px}{z}\right)\left(1 + \frac{qx}{z}\right) + \frac{1}{2}r^2_{bis}\right]\right\}$$

where p, q and z are the values used in the original formula and x is the abscissal distance from area q and is obtained from Table IV.

III. Both Variables Are Non-Quantitative

1. *Both Sets of Variables Occur in Several Classes*

The coefficient of mean square contingency, C, is a measure of relationship between variables expressed in qualitative categories, e.g., colors, temperaments, health, religious preferences, etc. When both characters are normally distributed and the number of categories is large, the value of C approaches the product moment r as a limit. The amount of relationship is assumed to be a function of the correspondence between actual cell frequencies and those to be expected in a pure chance distribution. The formula for determining the coefficient is:

$$C = \sqrt{\frac{U - 1}{U}}$$

where U is obtained as follows: Using, for example, a 3×3 fold distribution table and the notation a, a', a'', etc., for cell frequencies, A, B, C, etc., for totals of rows and columns, the table has the form

a	a'	a''	A
b	b'	b''	B
c	c'	c''	C
D	E	F	N

The value U may be found from the formula:

$$U = \frac{1}{D}\left[\frac{a^2}{A} + \frac{b^2}{B} + \frac{c^2}{C}\right] + \frac{1}{E}\left[\frac{a'^2}{A} + \frac{b'^2}{B} + \frac{c'^2}{C}\right] + \frac{1}{F}\left[\frac{a''^2}{A} + \frac{b''^2}{B} + \frac{c''^2}{C}\right].$$

For 4×4 fold, 5×5 fold and larger tables, the formula is extended to include a corresponding number of bracketed terms. In computing U, it is convenient to arrange the bracketed terms in a vertical column. The method is illus-

trated with eye-color data for offspring and great-grandparents:

Eye color of offspring

Eye color of great-grandparents	Blue	Gray Blue-green	Dark gray Hazel	Light brown Brown	Dark brown Black	Totals
Blue	192	103	81	45	48	469
Gray Blue-green	70	85	52	22	27	256
Dark gray Hazel	36	21	27	9	16	109
Light brown Brown	43	27	17	33	24	144
Dark brown Black	25	33	36	11	30	135
Totals	366	269	213	120	145	1113

$$C = \sqrt{\frac{1.07299 - 1}{1.07299}} = 0.2608.$$

Computation of C

$$\frac{1}{366}\left[\frac{192^2}{469} + \frac{70^2}{256} + \frac{36^2}{109} + \frac{43^2}{144} + \frac{25^2}{135}\right] = 0.34727$$

$$\frac{1}{269}\left[\frac{103^2}{469} + \frac{85^2}{256} + \frac{21^2}{109} + \frac{27^2}{144} + \frac{33^2}{135}\right] = 0.25286$$

$$\frac{1}{213}\left[\frac{81^2}{469} + \frac{52^2}{256} + \frac{27^2}{109} + \frac{17^2}{144} + \frac{36^2}{135}\right] = 0.20116$$

$$\frac{1}{120}\left[\frac{45^2}{469} + \frac{22^2}{256} + \frac{9^2}{109} + \frac{33^2}{144} + \frac{11^2}{135}\right] = 0.12842$$

$$\frac{1}{145}\left[\frac{48^2}{469} + \frac{27^2}{256} + \frac{16^2}{109} + \frac{24^2}{144} + \frac{30^2}{135}\right] = 0.14328$$

$$U = 1.07299$$

The coefficient of contingency, C, does not have negative values, but varies between 0 and 1.0. Its value is limited by

the number of categories used in the distribution, and Yule (1932, p. 66) gives for various small numbers of categories the limiting values of C.

Number of categories each way	2	3	4	5	6	7	8	9	10
C limit	0.707	0.816	0.866	0.894	0.913	0.926	0.935	0.943	0.949

A correction for broad grouping (few classes) has been given by Pearson (1913):

$$cC = \frac{C}{r_{xc}r_{yc}}$$

where r_{xc} and r_{yc} = correlation of variable with class value. Using the convenient notation of our 3×3 fold table, these r's may be obtained from the formulas:

$$r_{xc} = \sqrt{\frac{N}{D}(z_0 - z_1)^2 + \frac{N}{E}(z_1 - z_2)^2 + \frac{N}{F}(z_2 - z_3)^2 \ldots \text{etc.}}$$

$$r_{yc} = \sqrt{\frac{N}{A}(z'_0 - z'_1)^2 + \frac{N}{B}(z'_1 - z'_2)^2 + \frac{N}{C}(z'_2 - z'_3)^2 \ldots \text{etc.}}$$

where z_0 = ordinate at zero area. z'_0 = ordinate at zero area each = 0.

z_1 = ordinate at area $\frac{D}{N}$. z'_1 = ordinate at area $\frac{A}{N}$.

z_2 = ordinate at area $\frac{E}{N}$. z'_2 = ordinate at area $\frac{B}{N}$.

z_3 = ordinate at area $\frac{F}{N}$. z'_3 = ordinate at area $\frac{C}{N}$.

The ordinates may be read from Table III*a*.

The correction becomes less important with finer grouping and is generally omitted with 6×6 fold or finer classifications.

The probable error of C may be found from the formula:

$$PE_c = \frac{0.6745}{\sqrt{N}} \cdot \left[\frac{\frac{\psi^3}{\phi^2} + 1 - \phi^2}{(1 + \phi^2)^3}\right]^{1/2}$$

where $\phi^2 = U - 1$; and

$$\psi^3 = \frac{1}{N}\left\{\left[\frac{\left(a - \frac{AD}{N}\right)^3}{\left(\frac{AD}{N}\right)^2}\right] + \left[\frac{\left(b - \frac{BD}{N}\right)^3}{\left(\frac{BD}{N}\right)^2}\right] + \ldots \text{ etc. for each cell}\right\}$$

using the notation of the 3×3 fold table given above.

2. *One of the Two Sets of Variables Occurs in Several Classes*

Biserial eta (*bis* η) is a method of determining correlation where one variable is given by alternative and the other by multiple categories (Pearson, 1910). Regression is not assumed to be linear, and the categories are not quantitative. But the alternative category is assumed to have the Gaussian distribution.

$$bis\,\eta_{xy} = \sqrt{\frac{K^2 - x^2}{1 + K^2}}$$

where x is the alternative variable and y the multiple variable.

$$K^2 = \frac{1}{N}\Sigma(n_y x^2{}_y).$$

x_y is the proportion of the total area (as determined from Table IV) lying above the point of dichotomy.

n_y, the number of cases in a y category.

N, the number of cases in the total distribution.

Example. Offspring

		VS	S	M	F	VF	Total
Parents	Slender	5	140	192	41	11	389
	Not slender	4	148	582	244	74	1052
	Totals	9	288	774	285	85	1441

V, very; *S*, slender; *M*, medium; *F*, fleshy.

$q' = 0.55556 \quad 0.48611 \quad 0.24806 \quad 0.14386 \quad 0.12941$

$x_y = 0.1397 \quad 0.0348 \quad 0.6806 \quad 1.0632 \quad 1.1292$

$n_y x_y^2 = 0.1756 + 0.3488 + 358.5295 + 322.1623 + 108.3829 = 789.5991$

$$K^2 = \frac{789.5991}{1441} = 0.54795. \qquad x = 0.61282.$$

$$p = \frac{1052}{1441} = 0.7300. \qquad \text{bis}\eta_{xy} = \sqrt{\frac{0.54795 - 0.61282^2}{1 + 0.54795}} = 0.3337.$$

$$q = \frac{389}{1441} = 0.2700.$$

To obtain each x_y enter body of Table IV with value for $0.5 - q'$. If $q' > 0.5$, read x/σ value for $q' - 0.5$ and prefix a minus sign.

The standard error for biserial eta less than 0.500 is obtained by the formula:

$$SE_{bis\eta} = \frac{1 - \eta^2}{\sqrt{N}} \sqrt{\frac{pq}{y^2} + \frac{2px^2}{(1 + x^2)^2}}.$$

The y or ordinate value is obtained from Table IIIa.

Enter the table with $x/\sigma = 0.6128$ and read $y = 0.3306$.

Thus the standard error for the problem above is:

$$SE_{bis\eta} = \frac{1 - 0.3337^2}{\sqrt{1441}} \sqrt{\frac{0.2700 \times 0.7300}{0.3306^2} + \frac{2 \times 0.7300 \times 0.6128^2}{(1 + 0.6128^2)^2}}$$
$$= 0.0341.$$

For eta values greater than 0.50, a longer formula for the standard error is needed (Pearson, 1917).

3. *Each Set of Variables Occurs in Two Classes*

Tetrachoric Correlation. This is a method of expressing correlation quantitatively when the variables can not be so expressed, as, for example, in the case of effectiveness of vaccination. Strictly, this method assumes normal variation in variables, but it can be employed generally, in default of a better method, with fairly accurate results.

The prime requisite is that the qualities to be compared shall be separable into two grades, an upper and a lower. For example, in the case of the result of vaccination: on the one hand, either presence or absence of a scar; on the other, either recovery or death. As either of the second pair may occur with either of the first pair, four classes, a, b, c, d, will be formed altogether, and a correlation surface like the following may be made:

	$-y$			
	a	b	$a + b$	
$-x$				x
	c	d	$c + d$	
	$a + c$	$b + d$	N	
	y			

The surface should be so arranged that $a + b > c + d$ and $a + c > b + d$.

The axes y, $-y$ and x, $-x$ probably do not coincide with the axes y and x passing through the "origin" of the correlation surface, but may be regarded as situated from those axes at the respective distances h and k. These values may be found from the formulæ

$$\frac{(a + c) - (b + d)}{N} = \sqrt{\frac{2}{\pi}} \int_0^h e^{-\frac{1}{2}x^2}\,dx;$$

$$\frac{(a + b) - (c + d)}{N} = \sqrt{\frac{2}{\pi}} \int_0^k e^{-\frac{1}{2}y^2}\,dy.$$

a, b, c, and d being known, h and k are found from Table IV. Then

$$H = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2} \quad \text{and} \quad K = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2},$$

of which the values may be looked up in Table III, or, better, their product may be calculated by logarithms as follows:

$$\log HK = 9.201820 - N\left[\log \frac{h^2 + k^2}{2} + 9.637784\right].$$

Find also $\log hk$, h^2 and k^2. To find r solve the following equation to as many terms as may be necessary:

$$\frac{ad - bc}{N^2HK} = r + \frac{hk}{2} r^2 + \frac{1}{6}(h^2 - 1)(k^2 - 1)r^3$$
$$+ \frac{1}{24} hk(h^2 - 3)(k^2 - 3)r^4$$

$$+\frac{1}{120}(h^4 - 6h^2 + 3)\,(k^4 - 6k^2 + 3)r^5$$

$$+\frac{1}{720}hk(h^4 - 10h^2 + 15)(k^4 - 10k^2 + 15)r^6 + \text{etc.}$$

This gives us a numerical equation of the nth degree which can be solved by ordinary algebraic methods, using Sturm's functions and Horner's method. Or it can be solved by successive approximations as follows: The first approximation is made by neglecting all powers of r above the second and solving the quadratic (remembering that, if $ax^2+bx+c=0$, $x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$), and taking the positive root. Substitute this value in the whole equation to the fourth power for $f(r)$, and in the first derivative of the same equation for $f'(r)$ (remembering that the first derivative of $f(x)$ is obtained by multiplying each term in $f(x)$ by the exponent of x in that term and diminishing the exponent of x by 1). The correction $\frac{f(r)}{f'(r)}$ should be added to the value of r used in substituting. Repeat this process as often as the correction affects the fourth place of decimals, and go to r^5 if necessary.

Example. The eye colors of a certain set of people (see Biometrika, II, 2, pp. 237–240) and of their great-grandparents were found to be distributed as follows:

Offspring

Great-grandparents	1 Light blue	2 Blue—dark blue	3 Gray—blue-green	4 Dark gray—hazel	5 Light brown	6 Brown	7 Dark brown	8 Black	Totals
1. Light blue........	4	3	8	5	....	1	....	....	21
2. Blue—dark blue..	8	177	95	76	5	39	31	17	448
3. Gray—blue-green.	1	69	85	52	2	20	26	1	256
4. Dark gray—hazel.	6	30	21	27	2	7	15	1	109
5. Light brown......	....	4	....	....	....	....	....	....	4
6. Brown...........	2	37	27	17	3	30	20	4	140
7. Dark brown......	....	15	20	24	3	4	9	9	84
8. Black...........	....	10	13	12	2	2	7	5	51
Totals.........	21	345	269	213	17	103	108	37	1113

It was desired to determine the correlation between the eye color of the offspring and that of their great-grandparents. Clearly the ranges of the classes given above are not quantitatively equal nor determinable. Consequently a fourfold table was formed by dividing the population into those having eyes whose color was gray blue-green, or lighter, and those having dark gray, hazel, or darker eyes. This gives a good basis for calculation. If the dark gray and hazel eyes had been grouped with the lighter eyes it would have made quadrant a entirely too large; and there is nothing in the nature of the data that strongly favors one division more than another.

$$a_1 = \frac{725 - 388}{1113} = 0.302785$$

$$a_2 = \frac{635 - 478}{1113} = 0.141060$$

From Table IV:

Great-grandparents \ Offspring	1–3	4–8	Totals.
1–3	450	275	725
4–8	185	203	388
Totals	635	478	1113

$\frac{1}{2}a_1$	h (approx.)	$\frac{1}{2}a_2$	k (approx.)
0.151392		0.07053	
0.15173	0.390	0.07064	0.178
0.15136	0.389	0.07025	0.177
0.00037		0.00039	

$$h = 0.389 + (0.000032 \div 0.37) = 0.38909$$

$$k = 0.177 + (0.00028 \div 0.39) = 0.17772$$

$hk = 0.069150$; $\frac{1}{2}hk = 0.034575$.

$h^2 = 0.151392$; $k^2 = 0.031585$; $\dfrac{h^2 + k^2}{2} = 0.091489$.

$$0.224962 = r_t + 0.034575r_t^2 + \tfrac{1}{6}(h^2 - 1)(k^2 - 1)r_t^3 + \tfrac{1}{24}hk(h^2 - 3)(k^2 - 3)r_t^4 + \text{etc.}$$

Solving $0.034575r_t^2 + r_t - 0.224962 = 0$:

$$r_t = \frac{-1 \pm \sqrt{1 + 4(0.034575 \times 0.224962)}}{2(0.034575)}$$

$= 0.223225$ to first approximation.

$h^2 - 1 = 0.848608$; $k^2 - 1 = -0.968415$; coeff. $r_t^3 = 0.136967$.

$$\text{coeff. } r_t^4 = \frac{0.069150 \times 2.848608 \times 2.968415}{24} = 0.024363.$$

$$0.024363r_t^4 + 0.136967r_t^3 + 0.034575r_t^2 + r_t - 0.224962 = 0.$$

Applying Newton's approximation, we reach the result

$$r_t = 0.2217.$$

The probable error of tetrachoric r_t is given approximately by the formula:

$$PEr_t = 0.6745 \sqrt{\left[1 - r_t^2\right] \left[\left(\frac{1 - \sin^{-1} r_t}{90°}\right)^2\right]} \left\{\frac{\sqrt{(a + b)\,(a + c)\,(c + d)\,(b + d)}}{N^{2\prime} y y' \sqrt{N}}\right\}.$$

The first factor may be found in Table XVIII.

y and y' are found as follows: $\frac{a + b}{N} = \frac{725}{1113} = 0.6514$, the proportion of the cases in $a + b$. Since exactly half of the cases lie on each side of the mean in a normal distribution, the proportion of area beyond the mean will be $0.6514 - 0.5000 = 0.1514$. Since 0.1514 represents area, the $\frac{x}{\sigma}$ or abscissal distance is found in Table IV. The same procedure is gone through for $\frac{a + c}{N}$. Then enter Table III*a* and find the ordinates corresponding to the two abscissæ found in Table IV. These two ordinates are y and y', respectively.

$$PEr_t = \frac{0.6510 \sqrt{725 \times 635 \times 388 \times 478}}{(1113)^2 \times 0.3698 \times 0.3927 \times \sqrt{1113}} = 0.032.$$

B. INTERDEPENDENCE BETWEEN THREE OR MORE SETS OF VARIABLES

I. Coefficient of Partial Correlation

The coefficient of partial correlation is a measure of the net relationship between two variables when one or more other, more or less dependent, variables are held constant. For example, the correlation between stature and total heat production was found to be 0.622. Also, the correlations between weight and heat production, and weight and stature, were found to be 0.818 and 0.571 respectively. Now it is possible that a part or all of the correlation between stature and heat production is due to the correlation of stature with weight,

which in turn is correlated with heat production. That is to say, stature may be correlated with heat production through a third variable, in this case, weight.

1. *Computation of r*

In order to eliminate the influence of a varying weight for different individuals on the correlation between stature and heat production, one could compute the desired correlation by using only those individuals who had the same weight (within a small range). This correlation was found to be 0.197 for the 41 individuals whose weights ranged from 50 kilograms to 60 kilograms.

A better way of holding one variable constant and then finding the correlation between two other variables is offered by the following formula:

$$r_{12.3} = \frac{r_{12} - r_{13} \cdot r_{23}}{\sqrt{1 - r^2_{13}}\sqrt{1 - r^2_{23}}}$$

Letting stature = 1, heat production = 2; weight = 3, $r_{12.3}$ is a symbol which is read as follows: "The partial correlation between 1 and 2 with the influence of 3 held constant or eliminated." After assigning a number to each variable, as above, $r_{12.3}$ is said to be "the partial correlation between stature and heat production with the influence of weight eliminated or held constant."

Substituting in the formula

$$r_{12.3} = \frac{r_{12} - r_{13} \cdot r_{23}}{\sqrt{1 - r^2_{13}}\sqrt{1 - r^2_{23}}},$$

we have

$$r_{12.3} = \frac{0.622 - (0.571)\,(0.818)}{\sqrt{1 - (0.571)^2}\sqrt{1 - (0.818)^2}} = 0.328.$$

The correlation between stature and heat production with weight held constant is thus about one-half of the correlation where the individuals involved had different weights. This reduction means that part of the greater heat production of tall individuals is due to their greater weight. The value of 0.328 for $r_{12.3}$ indicates that there is a tendency for tall individuals to have a greater heat production than shorter ones even though they all have the same weight.

Eliminating the influences of weight on the correlation between stature and heat production by using only those individuals who have the same weight, or computing $r_{12.3}$ by the above formula, gives practically the same results. The difference obtained by the two methods in the above example is 0.131, which is not significant when its probable error is taken into consideration. The selection method involves a reduction in the number of individuals on which to base the correlation. On the other hand, the computation of $r_{12.3}$ affords a measure of relationship based on the weighted variability of all the arrays. It serves to hold variables constant without reducing the number of cases.

r_{12}, $r_{12.3}$, $r_{12.34}$, $r_{12.34\ldots n}$ are called correlations of the zero, first, second and nth orders, the order being determined by the number of subscripts to the right of the decimal point. The subscripts before and after the decimal point are called the primary and secondary subscripts respectively. In correlations the order of both the primary and secondary subscripts is indifferent. Thus, $r_{12.34} = r_{21.34} = r_{21.43} = r_{12.43}$.

To calculate partial correlations for any order it is only necessary to work out the equation:

$$r_{12.34\ldots n} = \frac{r_{12.34\ldots(n-1)} - r_{1n.34\ldots(n-1)}\, r_{2n.34\ldots(n-1)}}{\sqrt{1 - r^2_{1n.34\ldots(n-1)}}\,\sqrt{1 - r^2_{2n.34\ldots(n-1)}}}.$$

Thus:
$$r_{12.3} = \frac{r_{12} - r_{13}\cdot r_{23}}{\sqrt{1 - r^2_{13}}\,\sqrt{1 - r^2_{23}}};$$

also
$$r_{12.34} = \frac{r_{12.3} - r_{14.3}\cdot r_{24.3}}{\sqrt{1 - r^2_{14.3}}\,\sqrt{1 - r^2_{24.3}}}.$$

It has been stated that the order of both the primary and secondary subscripts is indifferent. Accordingly,

$$r_{12.34} = \frac{r_{12.3} - r_{14.3}\cdot r_{24.3}}{\sqrt{1 - r^2_{14.3}}\,\sqrt{1 - r^2_{24.3}}} = r_{12.43}$$

$$= \frac{r_{12.4} - r_{13.4}\cdot r_{23.4}}{\sqrt{1 - r^2_{13.4}}\,\sqrt{1 - r^2_{23.4}}}.$$

Rewriting the equation in this way is an excellent means of checking the work.

The following data were taken from a study by Harris and Benedict (1919) on basal metabolism in man. These data will be used to illustrate the computation of partial correlations, partial sigmas, coefficients of net regression, regression equations and multiple correlations.

Variable	Mean	σ
1. Total heat production in calories per day	1631.74	204.66
2. Stature in centimeters	172.96	7.59
3. Weight in kilos	64.10	10.30
4. Age in years	26.88	8.77

Correlations

$$r_{12} = 0.6149. \qquad r_{23} = 0.5725.$$
$$r_{13} = 0.7960. \qquad r_{24} = -0.1154.$$
$$r_{14} = -0.3062. \qquad r_{34} = 0.0067.$$

To find the correlation between total heat production and stature with the effects of age and weight held constant, it is necessary to solve the equation:

$$r_{12.34} = \frac{r_{12.3} - r_{14.3} \cdot r_{24.3}}{\sqrt{1 - r^2_{14.3}}\sqrt{1 - r^2_{24.3}}}$$

where $r_{12.3} = \dfrac{r_{12} - r_{13} \cdot r_{23}}{\sqrt{1 - r^2_{13}}\ \sqrt{1 - r^2_{23}}}$

$$= \frac{0.6149 - (0.7960)(0.5725)}{\sqrt{1 - 0.7960^2}\sqrt{1 - 0.5725^2}} = 0.32077.$$

$$r_{14.3} = \frac{r_{14} - r_{13} \cdot r_{34}}{\sqrt{1 - r^2_{13}}\sqrt{1 - r^2_{34}}}$$

$$= \frac{-0.3062 - (0.7960)(0.0067)}{\sqrt{1 - 0.7960^2}\sqrt{1 - 0.0067^2}} = -0.51469.$$

$$r_{24.3} = \frac{r_{24} - r_{23} \cdot r_{34}}{\sqrt{1 - r^2_{23}}\sqrt{1 - r^2_{34}}}$$

$$= \frac{-0.1154 - (0.5725)(0.0067)}{\sqrt{1 - 0.5725^2}\sqrt{1 - 0.0067^2}} = -0.14543.$$

$$\text{Thus, } r_{12.34} = \frac{0.32077 - (-0.51469)(-0.14543)}{\sqrt{1-(-0.51469)^2}\sqrt{1-(-0.14543)^2}} = 0.28991$$

or $r_{12.34} = 0.290.$*

Conclusion: In this sample, about half of the correlation between total heat production and stature is due to the varying ages and weights of the individuals involved ($r_{12.34}$ is about half of r_{12}). Nevertheless, there is a tendency for the taller individuals to have a greater total heat production than shorter ones, even though all the individuals involved have the same age and weight.

2. *Partial Sigmas*

In the discussion of correlation involving two variables it was pointed out that the standard error of estimate is a measure of the variability about the regression line, and that in a particular problem, the smaller the standard error of estimate, the more reliable a prediction based upon the regression equation would become. Partial sigmas may also be interpreted in exactly the same way, only in this case the variability is measured about a regression line describing the relationship between one variable, x_1, and a series of other variables $x_2, x_3, x_4, \ldots, x_n$. In other words, a partial sigma is a measure of the variability remaining in x_1, after the variability due to $x_2, x_3, \ldots, x_n$ has been eliminated.

A partial sigma of order n may be written as $\sigma_{1.234\ldots n}$. The subscripts to the right of the decimal point indicate the variables whose effect on the variability of x_1 has been eliminated.

A partial sigma of order n may be computed as follows:

$$\sigma_{1.234\ldots n} = \sigma_1 \sqrt{1 - r^2_{12}}\sqrt{1 - r^2_{13.2}}\sqrt{1 - r^2_{14.23\ldots n}} \sqrt{1 - r^2_{1n.23\ldots(n-1)}}$$

thus, $\sigma_{1.23} = \sigma_1 \sqrt{1 - r^2_{12}}\sqrt{1 - r^2_{13.2}}$;

$$\sigma_{1.234} = \sigma_1 \sqrt{1 - r^2_{12}}\sqrt{1 - r^2_{13.2}}\sqrt{1 - r^2_{14.23}}.$$

* In the calculations it is necessary to carry out the computations to more places than is desired in the final answer in order to avoid cumulative errors due to the rounding off of numbers.

Since the subscripts to the right of the decimal point merely indicate what variables are held constant their arrangement is immaterial as far as the numerical value of the partial sigma is concerned. Thus $\sigma_{1.23} = \sigma_{1.32} = \sigma_1 \sqrt{1 - r^2_{13}} \sqrt{1 - r^2_{12.3}}$. Recomputing $\sigma_{1.23}$ in this manner gives a convenient check on the calculation.

For the metabolism problem the partial sigmas are as follows:

$$\sigma_{1.23} = \sigma_1 \sqrt{1 - r^2_{13}} \sqrt{1 - r^2_{12.3}}$$

$$= 204.66 \sqrt{1 - (0.7960)^2} \sqrt{1 - (0.3208)^2} = 117.332;$$

$$\sigma_{2.13} = \sigma_2 \sqrt{1 - r^2_{23}} \sqrt{1 - r^2_{12.3}}$$

$$= 7.59 \sqrt{1 - (0.5725)^2} \sqrt{1 - (0.3208)^2} = 5.894;$$

$$\sigma_{3.12} = \sigma_3 \sqrt{1 - r^2_{13}} \sqrt{1 - r^2_{23.1}}$$

$$= 10.30 \sqrt{1 - (0.7960)^2} \sqrt{1 - (0.1740)^2} = 6.139.$$

$\sigma_{1.23}$ is the variability of the total heat production after the variability due to the different statures and weights of the individuals involved has been removed. $\sigma_{2.13}$ and $\sigma_{3.12}$ may be interpreted in a similar way.

The regression coefficients are calculated by the formula:

$$b^*_{12.34 \ldots n} = r_{12.34 \ldots n} \frac{\sigma_{1.234 \ldots n}}{\sigma_{2.134 \ldots n}}.$$

Thus,

$$b_{12.3} = r_{12.3} \frac{\sigma_{1.23}}{\sigma_{2.13}};$$

$$b_{13.2} = r_{13.2} \frac{\sigma_{1.23}}{\sigma_{3.12}};$$

$$b_{21.3} = r_{12.3} \frac{\sigma_{2.13}}{\sigma_{1.23}}.$$

---

* In calculating partial correlation coefficients it was stated that $r_{12.34} = r_{21.34}$. In regression coefficients, however, $b_{12.34}$ does not equal $b_{21.34}$, since $b_{12.34}$ is the slope of X_1 on X_2 and $b_{21.34}$ is the slope of X_2 on X_1, it being understood that the effects of X_3 and X_4 have been eliminated before these slopes were measured.

3. *Regression Equations*

The equation of net regression of X_1 on X_2, $X_3 \ldots X_n$ gives a way of estimating X_1 from X_2, X_3 . . ., X_n, provided certain relationships are known. Thus the equation to predict X_1 from X_2 and X_3 is

$$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3.$$

Similarly to predict X_1 from X_2, X_3 and X_4

$$X_1 = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4,$$

and, in general,

$$X_1 = a_{1.234\ldots n} + b_{12.34\ldots n}X_2 + b_{13.24\ldots n}X_3 + \ldots + b_{1n.23\ldots(n-1)}X_n$$

where X_1 = the variable to be predicted.

$X_2, X_3, \ldots, X_n$ = the variables used in predicting X_1.

The constant term

$$a_{1.234\ldots n} = M_1 - b_{12.34\ldots n}X_2 - b_{13.24\ldots n}X_3 - \ldots - b_{1n.23\ldots(n-1)}X_3.$$

Thus, $a_{1.23} = M_1 - b_{12.3}X_2 - b_{13.2}X_3.$

$$a_{1.234} = M_1 - b_{12.34}X_2 - b_{13.24}X_3 - b_{14.23}X_4.$$

$b_{12.34\ldots n}$ = the net regression coefficient of X_1 on X_2, when the effect of the variables $X_3, X_4, \ldots, X_n$ is held constant. It is the weight to be given to X_2 in predicting X_1 from $X_2, X_3, X_4, \ldots X_n$. Similarly $b_{13.24\ldots n}$ is the weight to be given to X_3 in predicting X_1 from $X_2, X_3, X_4, \ldots, X_n$.

In the present problem to predict an individual's " total heat production " from a knowledge of his stature and weight, it is necessary to use a regression equation involving three variables, namely,

$$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3$$

where X_1 is the predicted " total heat production " for an individual of X_2 height and X_3 weight. And where

$$b_{12.3} = r_{12.3}\frac{\sigma_{1.23}}{\sigma_{2.13}} = 0.3208\,\frac{117.332}{5.894} = 6.39;$$

$$b_{13.2} = r_{13.2}\frac{\sigma_{1.23}}{\sigma_{3.12}} = 0.6866\,\frac{117.332}{6.139} = 13.12;$$

$$a_{1.23} = M_1 - b_{12.3}X_2 - b_{13.2}X_3$$

$$= 1631.74 - 1105.21 - 840.99 = -314.46;$$

$$X_1 = -\,314.46 + 6.39X_2 + 13.12X_3.$$

II. Multiple Correlation

The aim of multiple correlation is to test how closely it is possible to predict a variable, say X_1, from several other variables such as X_2, X_3, . . ., X_n. It is desirable to know whether the equation expressing the relationship between " total heat production " and stature and weight gives results which check up closely with the actual measurements.

Thus, if the predicted "total heat productions" are correlated with the actual values, a certain correlation will be found, and it is obvious that the more closely the predicted and the actual values agree, the higher will this correlation be.

A correlation of this type is called a multiple correlation, since a variable, say X_1, is being correlated with several other variables. The symbol for multiple correlation is $R_{1.234\ldots n}$. The subscript to the left of the decimal point indicates the variable that is being predicted from the variables X_2, X_3, X_4, . . ., X_n. The order in which these latter variables are written is immaterial.

Thus

$$R_{1.234\ldots n} = R_{1.324\ldots n}.$$

Multiple correlation coefficients may be calculated by the formula:

$$R_{1.234\ldots n} = \sqrt{1 - \frac{\sigma^2_{1.234\ldots n}}{\sigma_1^2}}$$

or from

$$R_{1.234\ldots n} =$$

$$\sqrt{1-(1-r^2_{12})(1-r^2_{13.2})(1-r^2_{14.23})\ldots(1-r^2_{1n.23\ldots(n-1)})}.$$

Since the order of the subscripts to the right of the decimal point is immaterial

$$R_{1.234\ldots n} = \sqrt{1-(1-r^2_{13})(1-r^2_{12.3})(1-r^2_{14.23})\ldots(1-r^2_{1n.23\ldots(n-1)})}.$$

Thus in the present problem the multiple correlation between "total heat production" and stature and weight is

$$R_{1.23} = \sqrt{1-\frac{\sigma^2_{1.23}}{\sigma_1^2}} = \sqrt{1-\left(\frac{117.332}{204.66}\right)^2} = 0.819$$

or

$$R_{1.23} = \sqrt{1-(1-r^2_{13})(1-r^2_{12.3})}$$
$$= \sqrt{1-(1-0.7960^2)(1-0.3208^2)} = 0.819.$$

It will be noted that predicting "total heat production" from weight gives results practically as good as estimating it from both stature and weight, since the respective correlations are 0.796 and 0.819.

III. The Coefficient of Part Correlation

The coefficient of part correlation (Smith and Ezekiel, 1926) measures the correlation between the dependent factor (after removing the net variations found to be associated with the remaining independent factors) and the particular independent factor to be considered. For example, if farm income 1 be dependent on (and the sum of) independent variables $2 + 3 + 4$, then if the terms 3 and 4 be subtracted from 1 in each case, it is possible to correlate this reduced 1 with the 2 part factor. Such an operation is designated as ${}_{12}r_{34}$, the subscript to the left of r indicating the dependent variable and the independent variable whose effect is being measured, while the subscripts to the right indicate the independent variables whose effects are removed.

This coefficient differs from the partial correlation coefficient in that all the original variation is left in the independent variable (2) and only the dependent variable (1) is adjusted. Formula:

$${}_{12}r_{34} = \sqrt{\frac{b^2_{12.34}\sigma_2^2}{b^2_{12.34}\sigma_2^2 + \sigma_1^2(1-R^2_{1.234})}} \quad \text{(coefficient of part correlation)}$$

where ${}_{12}r_{34}$ = the coefficient of part correlation.
$b_{12.34}$ = the partial regression coefficient, given by

$$r_{12.34}\frac{\sigma_{1.234}}{\sigma_{2.134}}.$$

$R_{1.234}$ = the multiple correlation coefficient.
σ_1, σ_2 = standard deviations of distributions.

Applying the formula to the problem of page 111 gives

$$ {}_{12}r_{34} = \sqrt{\frac{(6.39)^2(7.59)^2}{(6.39)^2(7.59)^2 + 204.66^2(1-0.819^2)}} = 0.382$$

Note that at page 111 $r_{12} = 0.615$ and $r_{12.3} = 0.321$ Further discussion is given by Ezekiel (1930, p. 182).

IV. Tetrad Difference

The tetrad difference idea was first applied in the attempt to prove the Spearman g theory (Spearman, 1927). It can be proved (statistically) that if four variables have one, and only one, common factor running through them, then every tetrad difference involving the r's of these four variables will be equal to 0. Tetrad differences are defined as follows:

$$t_{1234} = r_{12}r_{34} - r_{13}r_{24}.$$
$$t_{1243} = r_{12}r_{34} - r_{14}r_{23}.$$
$$t_{1342} = r_{13}r_{24} - r_{14}r_{23}.$$
$$t_{1324} = r_{13}r_{24} - r_{12}r_{34}.$$
$$t_{1423} = r_{14}r_{23} - r_{12}r_{34}.$$
$$t_{1432} = r_{14}r_{23} - r_{13}r_{24}.$$

Illustration: Coefficients of correlation between parts of an intelligence test.

	Opposites	Completion	Memory	Discrimination
Completion	0.80			
Memory	0.60	0.48		
Discrimination	0.30	0.24	0.18	
Cancellation	0.30	0.24	0.18	0.09

Let o = opposites. Then $r_{oc} = 0.80$.
c = completion. $r_{md} = 0.18$.
m = memory. $r_{om} = 0.60$.
d = discrimination. $r_{cd} = 0.24$.

Since $t_{1234} = r_{12}r_{34} - r_{13}r_{24}$;
then $t_{ocmd} = r_{oc}r_{md} - r_{om}r_{cd}$.

Substituting, $t_{ocmd} = 0.80 \times 0.18 - 0.60 \times 0.24$
$= 0.144 - 0.144$
$= 0.0$.

Any other tetrad equation using the above table of coefficients will be 0. If every tetrad difference equals 0, then the variables may be thought of as having one general factor and no group factors. In practice, one finds the distribution of obtained tetrad differences and computes the σ of the distribution. If this σ is no greater than the variability to be expected theoretically (Lexian ratio may serve as a measure), then one and only one general factor is common to all variables. (Of course each variable has a specific factor also.) For more complete account see T. L. Kelley (1928, p. 47). See also R. Pintner (1931.)

CHAPTER VI

HEREDITY

Mendelian Analysis

The study of heredity in man is especially difficult owing to the small size of fraternities. It consequently requires special methods.

The Family Method. Factorial analysis of traits arising from recessive genes meets with the difficulty of small families. The simplest recessives appear in only 1 child out of 4, and in families of even as many as 4 children, derived from 2 heterozygous parents, it will often happen that the expected recessive will not occur, even though (since the parents are both heterozygotes) the sibship potentially contains one.

Assume a group of heterozygous (DR) parents each producing 4 children. The probability of occurrence or association of the dominant and recessive phenotypes is obtained by the expansion of the binomial $(\frac{3}{4} + \frac{1}{4})^4$, as follows:

$$(\tfrac{3}{4})^4 + 4(\tfrac{1}{4})\cdot(\tfrac{3}{4})^3 + 6(\tfrac{1}{4})^2\cdot(\tfrac{3}{4})^2 + 4(\tfrac{1}{4})^3\cdot(\tfrac{3}{4}) + (\tfrac{1}{4})^4$$

or, in 256 families:

81, 108, 54, 12, 1

children are in the respective terms of the binomial formula, with proportion of recessives (R) and dominants (D) as follows:

$0R4D$, $1R3D$, $2R2D$, $3R1D$, $4R0D$.

That is to say, 81 families out of 256 will not show their potential recessives, and we can not say of such families whether or not both the parents are actually heterozygous.

Since in the above families recessives occur once in 108 times, twice in 54 times, thrice in 12 times and in four times once, or a total of 256, the observation of these families would

lead to the false conclusion that the incidence of the recessive trait was 256 out of 4×175, or 700, individuals instead of 4×256 or 1024 individuals. Thus a false ratio of 36.6 per cent instead of the true ratio of 25 per cent would be obtained.

It follows that with human families expectation in such matings is no longer 25 per cent, or in the case of back crossing to a phenotypically recessive parent *RR* 50 per cent, but a larger proportion, depending on the average number of children in the fraternities studied. In fraternities of 2 the ratios that will probably be found are 57 per cent and 67 per cent respectively; in fraternities of 5 children 33 per cent and 52 per cent; in fraternities of 10 children 26 per cent and 50 per cent respectively. The general formula for finding the unit recessive ratio that we may expect to find in $DR \times DR$ matings is $\frac{p}{1 - q^s}$, where p is the theoretical percentage, s the number of children in the fraternity and $q = 100 - p$. The charts (Figs. 23, 24) from Koller (1931) will be found useful in determining the proportion of unit recessives occurring in small fraternities, and the curve $DR \times DR$ can also be used to determine prediction in respect to very rare recessive sex-linked traits where the affected progeny are all males. But if the trait is not very rare, matings of affected males with female carriers may occur, in which case half of the offspring are recessives and curve $DR \times RR$ may be used to determine prediction. In the particular instance of hemophilia no well-authenticated cases of affected females are known.

In dealing with a pool of families from heterozygous parents and containing at least one recessive offspring let t be the observed number of recessives, c the maximum number of children in any family. Then if one is actually dealing with a recessive character, which has been observed in a population $\sum_{s=1}^{s=c} t$ times, where $t = \frac{pns}{1 - q^2}$

$$\frac{\sum_{s=1}^{s=c} t}{\sum_{s=1}^{s=c} \frac{ns}{1 - q^s}} = 0.25$$

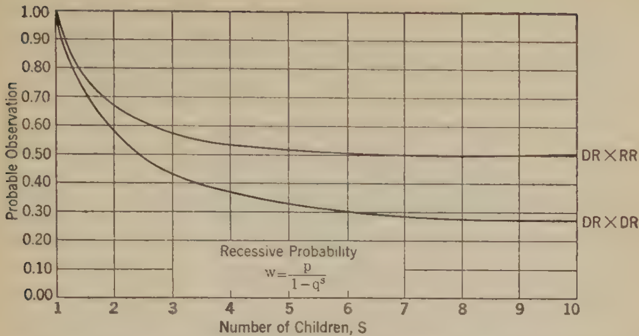

FIG. 23.—To show expected proportion of recessive children, in fraternities of s children, that will be found in DR × DR and DR × RR matings respectively. Where the families are very large, 50% and 25% recessives will be expected as indicated at the extreme right of the figure, where the DR × RR line quite touches 0.50 and the DR × DR line nearly touches 0.25. But as the number of children becomes smaller (toward the left) the expectation-proportion of recessive children rises (in the families selected because they show recessive children) until at $s = 2$ the ratios are .67 and .57 respectively. The curves are derived from the equation (in which p is the theoretical Mendel number and $q = 1 - p$) and enable one to read off the percentage of expected recessives in families of 1 to 10 children

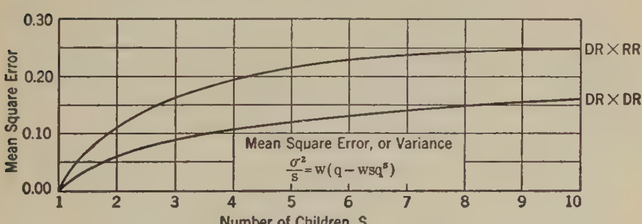

FIG. 24.—Variability in terms of variance $\left(\frac{\sigma^2}{s}\right)$ of the distribution of recessive children of DR × DR and DR × RR matings, in families where at least 1 child (of a fraternity of s children) is a recessive. The curves are drawn from the equation in which w is the theoretical probability of recessive children, as altered by the selection of affecte

If this equation is satisfied we may conclude as probable that a single pair of allelomorphs with dominance is involved.

Standard Error of Above Determination. Compare Σt with $\Sigma \tau$, in which τ represents the expected number of recessives, calculated as follows:

$$\Sigma \tau = \sum \frac{ns(1 - q)}{1 - q^s}.$$

The difference between $\Sigma \tau$ and Σt will then have a standard error of

$$\text{S.E. dif.} = \sqrt{\Sigma nk},$$

where $$k = \frac{s(1 - q)}{(1 - q^s)^2} [sq^s(q - 1) + q(1 - q^s)].$$

Lenz (1929, p. 707) offers the generalization that if a rare mono-recessive trait shows itself (phenotypically) in $\frac{1}{n}$ of the population the frequency of the corresponding gene in the population is $\frac{1}{\sqrt{n}}$. Also in the parents and children of the carriers of a trait the trait will appear phenotypically with a frequency of about $\frac{1}{\sqrt{n}}$; for all the parents and all the children of a carrier of the trait have received the appropriate gene at least once and another gene of the same kind will tend to coincide in fertilization in accordance with the general frequency, $\frac{1}{\sqrt{n}}$, of the gene in the population.

The Mass Method. This has been developed by Snyder (1934) for cases of a single factor for a unit character.

Type 1. *Without Dominance.* There is involved a pair of allelomorphs C and c. The heterozygote Cc is phenotypically recognizable. Three classes are recognizable in the population: the dominant homozygotes DD and the heterozygotes DR, and the recessive homozygotes RR. Let q = frequency of D allelomorphs, p frequency of R; then $p + q = 1$. The three phenotypes dependent on the genes CC, Cc and cc, respectively, will occur in the population in the proportion q^2, $2pq$ and p^2.

Hence

$$q = \sqrt{\text{proportion of } DD \text{ individuals.}}$$

$$p = \sqrt{\text{proportion of } RR \text{ individuals.}}$$

If the proportion of a genotype in the general population be indicated by a superimposed line, then $\overline{DD}$ stands for the proportion of homozygous dominant individuals; $\overline{RR}$ for the proportion of recessive individuals. Then $\sqrt{\overline{DD}} + \sqrt{\overline{RR}} = 1$. The standard error of the deviation from unity is the reciprocal of $2\sqrt{N - 1}$, where N is the number of people in the sample examined. If the equation $p + q = 1$ is satisfied, probably inheritance is of Type 1.

Type 2. A Pair of Allelomorphic Genes, One of Which Is Dominant. Call them D and R, D being dominant over R. Then if q = frequency of D and p = frequency of R, again $p + q = 1$. The dominant individuals will occur in the general population with the frequency $q^2 + 2pq$, and the recessive with frequency p^2. Then $p = \sqrt{\overline{RR}}$. If L = proportion of recessive type expected in the offspring of random matings of dominants with dominants and M = proportion of recessives to be expected in the offspring of random matings of dominants and recessives, then $L = \left(\frac{p}{1+p}\right)^2$; $M = \frac{p}{1+p}$. The observed proportions of recessives in the families studied may thus be compared with the expected proportions and if the difference between observed and calculated results are non-significant the hypothesis of a single pair of allelomorphs with dominance is satisfied (Table XXII). The standard

error of L is: $\mp \dfrac{1 - \sqrt{b}}{(1 + \sqrt{b})^2}\sqrt{\dfrac{b}{N(1 - b)}}$;

of M is $\mp \frac{1}{2}\dfrac{(1 - \sqrt{b})}{1 + \sqrt{b}}\sqrt{\dfrac{1}{N(1 - b)}}$,

where b is taken as the observed value of the proportion of recessives ($\overline{dd}$) in the general population, and N is the total number of individuals studied in the population.

Type 3. Case of Character Dependent upon Multiple Allelomorphs, A, a', a. Let allelomorphs A and a' be dominant

to a, but the heterozygote Aa' is phenotypically recognizable. Then four kinds of phenotypes occur in the population: those that show both dominant factors; those that show only one; those that show only the other; those that show neither dominant. Let q = frequency of A allelomorph; p = frequency of a', and r frequency of a. Then $p + q + r = 1$. Then individuals showing both dominants (Aa') will occur in the general population with frequency $2pq$; those showing one dominant (AA, Aa) with the frequency $q^2 + 2qr$; those showing the other dominant ($a'a'$, $a'a$) with the frequency $p^2 + 2pr$; and those showing neither dominant with the frequency r^2. The proportion of AA individuals is ($\overline{AA}$), of $a'a'$ individuals $\overline{a'a'}$, and of aa individuals is $\overline{aa}$. Then,

$$\sqrt{\overline{aa} + (\overline{AA} + \overline{Aa})} + \sqrt{\overline{aa} + (\overline{a'a'} + \overline{a'a})} - \sqrt{\overline{aa}} = 1.$$

If this equation is satisfied, the hypothesis of multiple allelomorphs is justified.

Type 4. *Case of Character Dependent upon Two Pairs of Factors.* Assume a pair of coöperating allelomorphs A and a, and a second pair B and b; with complete dominance. Let q = frequency of A gene; p = frequency of a, r = frequency of B, and s = frequency of b. Then $p + q = 1$; $r + s = 1$, and

$$s^2 = \overline{aabb} + (\overline{AAbb} + \overline{Aabb});$$

$$p^2 = \overline{aabb} + (\overline{aaBB} + \overline{aaBb});$$

$$(1 - p^2)\,(1 - s^2) = \overline{AABB} + \overline{AABb} + \overline{AaBB} + \overline{AaBb}.$$

If this equation is valid the hypothesis of two pairs of independent factors is probably true.

Type 5. *Expectation of Affected Offspring When the Trait Depends upon the Occurrence of Two Independent Genes (Double Dominant) and the Trait Is Phenotypically Absent in Both Parents.* What is the expectation of affected offspring (double dominant) in matings when neither parent shows the dominant trait; and the dominant trait depends upon the concurrence of two dominant genes (double dominants) A and B which are located on different chromosomes and which have germ cells of frequency in the population indicated by l and m respectively?

For a family of s members this is given by the formula:

$$\frac{l^2}{(l-2)^2}s + \frac{4l(1-l)}{(l-2)^2}\cdot\frac{0.5s}{1-0.5^s} + \frac{4(1-l)^2}{(l-2)^2}\cdot\frac{0.25s}{1-0.75^s}.$$

This expectation is close to the expectation for recessive offspring of two normal parents when the dominant genes are fairly rare (1 to 40 individuals, or less).

The proportion of affected individuals in a family where only one parent is phenotypically affected owing to the possession of the two dominant genes required can be computed from the tables given by Hogben (1933, pp. 84–87).

Type 6. *Case of Sex-influenced Traits.* Assume a pair of allelomorphs H and h, such that H is dominant in males but recessive in females. Let q = frequency of H and p = frequency of h. Then $p + q = 1$. Let $\overline{♀H}$ = proportion among women of women who show the trait represented by the H gene; $\overline{♂h}$ = the proportion among men of men who show the character represented by the h gene. Then $\sqrt{\overline{♀H}} + \sqrt{\overline{♂h}} = 1$. If this equation is satisfied the trait is inherited under this category.

Standard error of the deviation from unity:

$$\text{S. E. dev.} = \mp\tfrac{1}{2}\sqrt{\frac{1-B}{N_1} - \frac{(1-B)^2}{16B(N_1)^2} + \frac{1-D}{N^2} - \frac{(1-D)^2}{16D(N_2)^2}}$$

where $B = \overline{♀H}$, $D = \overline{♂h}$, N_1 = number of individuals from which the value of B was derived, and N_2 = number of individuals from which the value of D was derived.

Type 7. *Case of Sex-linked Traits.* Assume a pair of allelomorphs M and m, with M dominant to m, carried on the X-chromosomes. Let q = frequency of M gene, and p = frequency of m gene. Then $p + q = 1$. Designating the proportion among men of men who show the recessive trait as $\overline{♂m}$, and the proportion among women of women who show the recessive trait as $\overline{♀m}$, then $\sqrt{\overline{♀m}} = \overline{♂m}$. The standard error of this equation is:

$$\text{S.E.} \mp \sqrt{\frac{1-\text{B}}{4N} - \frac{(1-\text{B})^2}{64B(N_1)^2} + \frac{\text{D}(1-\text{D})}{N_2}}$$

where $B = \sqrt{\overline{\female m}}$, $D = \overline{\male m}$, N_1 = the number of individuals used in computing B, and N_2 = number of individuals used in computing D.

Determination of Linkage in Man. Assume two pairs of genes C and c, D and d, located on the same pair of chromosomes. A double heterozygote is then $CcDd$. Since these two genes are by hypothesis linked they may be written in parentheses; thus a double heterozygote involving linked factors is (CD) (cd) in the coupling phase, or (Cd) (cD) in the repulsion phase (Snyder, 1934).

Assume a series of family histories involving the cross $CcDd \times Ccdd$. The $CcDd$ parents would be recognized as being heterozygous for both genes by having produced some offspring of each recessive type. If the parents are both $Ccdd$ then cc offspring will have been produced. It is indeterminate whether the $CcDd$ parent was in the coupling or repulsion phase, i.e., whether the constitution is (CD) (cd) or (Cd) (cD), since the phenotype is the same in both. If we assume that the two phases are equally common it will not be necessary to differentiate them.

With the given constitution of the parents there are certain offspring called "determinate" that indicate whether the dominant genes are coupled or not. These show the recessive trait represented by c. Thus, if coupled, the offspring of genetic constitution $ccdd$ (without dominant gene) will be found and will be non-crossovers, while offspring of the formula $ccDd$ (showing one dominant) will be crossovers. If the dominant genes are not coupled in either parent the findings in the children are to be interpreted in opposite fashion. However, in any particular family one type of progeny will represent crossovers and the other type will represent non-crossovers.

In the long run, non-crossovers will be more abundant than crossovers. Consider a large number of families of which the genetic constitution of parents and children is known (with respect to two pairs of allelomorphs). Bring together similar crosses and list in a column headed U the "determinate" children of the more frequent type (who are probably non-crossovers), and in a column V the less frequent type (who are probably crossovers). The ratio of the frequency of V to the sum of $V + U$ gives Q, which is approximately the true pro-

portion of crossing over sought. By a refined method Wiener has prepared a closer approximation to true crossover values based on Q and number of children in a family. These are given in Table XX, derived from Wiener (1932, p. 343).

Example. Following Wiener. In the blood cells there are two substances called agglutinogens since they are capable of causing agglutination of the blood corpuscles under certain conditions of the blood. They are labelled A and B. In some blood the cells produce both agglutinogens AB; in other blood neither; such blood is called O, and is not further considered here.

In addition some blood contains agglutinogens M and N, either one, both or neither. A mating with the genetic formulas $AB ++ \times O +-$ means: mother has (symbols preceding the $\times$) both A and B and also M (first $+$) and N (second $+$); while the father (symbols following the $\times$) has neither A nor B; has M and lacks N. Assume the genotype of the mother in this case to be $(AM)(BN)$; her linked gametes are $(AM)(BN)$; her crossover gametes $(AN)(BM)$. The $O +-$ father produces OM and On gametes in equal numbers.

The linked zygotes are produced as below:

mating $\underbrace{AB++\times O+-}$

union of gametes

$$\begin{tikzcd} AM \arrow[r] \arrow[dr] & OM \\ BN \arrow[r] \arrow[ur] & On \end{tikzcd}$$

produced are

genotypes	phenotypes
$AMOM$	$A+-$
$AMOn$	$A+-$
$BNOM$	$B++$
$BNOn$	$B-+$

The crossed-over zygotes are produced as below:

mating $AB++\times O+-$

union of gametes

$$\begin{tikzcd} {}^*AN \arrow[r] \arrow[dr] & On \\ {}^*BM \arrow[r] \arrow[ur] & OM \end{tikzcd}$$

produced are

zygotes

genotypes	phenotypes
$^*ANOn$	$^*A-+$
$^*ANOM$	$A++$
$^*BMOn$	$B+-$
$^*BMOM$	$B+-$

Thus the linked types are $A+-$ and $B-+$, while the crossover types are $A-+$ and $B+-$. Progeny of type $A++$ and $B++$ might be either linked or crossovers and must therefore be classified as "indeterminate."

One makes out a table of analyses like that of the last paragraph for all the matings with which one is dealing [e.g., $AB ++\times O ++$; A (heterozygous) $++\times O -+$; A (heterozygous) $++\times B$ (heterozygous) $+-$; B (heterozygous) $++\times A$ (pure, p) $++$], and lists the genetic and phenotypic constitution of all kinds of determinate types of offspring and classifies them as linked or crossovers. The indeterminate may be neglected.

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* Crossed-over gametes or zygotes.

Thus from	linked offspring	crossover offspring
$AB ++ \times O\ \ ++$	$A\ \ +-\ B\ -+$	$A\ \ -+\ B\ +-$
$AB ++ \times Ap\ ++$	$AB\ -+\ A\ +-$	$AB\ +-\ A\ -+$

Then rule a number of columns, each column for a family of a certain number of offspring, e.g., 4. Rule 3 subcolumns, the first with the identification number of the family, the second for the number of children of the commoner type (call it U) and the third for the number of children of the less common type (call it V). If the number of children of the two types is the same enter this number in both columns. U represents the sum of the linked types for each column; V, the crossover types. $U + V =$ the total determined types for each column. $V/(U + V)$ gives the value of Q, the percentage value of V or of crossing over.

The theoretically expected value of Q for different numbers of children and for various crossover values (top line) has been computed by Wiener,* and the results of such computation are given in Table XX. If the computed Q for the given number of determinate children is less than the crossover value found in column 0.50 by a difference equal to more than 2 times the standard error as given in Table XXI, the traits in question are probably linked and the indicated crossing over occurs.

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* With the use of the formula

$$Q = pq + (pq)^2 + 2(pq)^3 + 5(pq)^4 + 14(pq)^5 + 42(pq)^6 + 132(pq)^7 + 429(pq)^8 + \ldots$$

where p represents the true linkage intensity and q the crossover frequency so that $p + q = 1$.

Let s equal the number of determinate children. To determine Q from this formula, $s/2$ terms are taken if s is even; $(s - 1)/2$ if s is odd. Thus if $s = 7$ and $p = 0.50$, $Q = (0.25) + (0.25)^2 + 2(0.25)^3 = 0.34375$.

CHAPTER VII

Special Topics

Growth

General growth in organisms is the phenomenon of increase in size either of the body as a whole or of multiplying cells. One must carefully avoid confusing growth of an aggregate (or average) of a sample on the one hand and growth of an individual on the other. The various formulæ that have been proposed to describe growth have mostly to do with averages and not individuals and hence have a statistical value rather than a biological interest.

a. "Autocatalytic": $y = (a + be^{-nt})^{-1}$. This is an S-shaped curve asymptotic to the X-axis at the lower end and to a parallel to the X-axis at the upper end, and lying at a distance from the X-axis of $k = \frac{1}{a}$. k is the final size. The curve has an inflection of the value at $\frac{1}{2}a$. b and n are constants depending on the starting point and slope of the curve. It is also called the logistic curve. This formula is suitable for populations, where t takes value from $-\infty$ to $+\infty$.

b. Gompertz: $y = ke^{-e^{a-bt}}$ or $\log\log\ (k/y) = a - bt$. This curve is not symmetric about the inflection. This has been usefully employed in computing mortality of life-insurance-policy holders.

c. Algebraic: $y = k(t/T)^3(2 - t/T)^3$. This curve rises as a cubic from $t = 0$ to $t = T$ (where it terminates). It has an inflection at $0.55T$, $y = 64k/125$.

For some mammals (though not for man) the curve of the most rapidly growing period from shortly after birth is given by $y/k = 1 - e^{-n(t-t^*)}$, where k and n are constants of final size and time involved in growth and t^* is the time of inflection of growth.

Law of Relative Growth. As proposed by Huxley (1932, pp. 6, 7) this is: $(dy/y)/(dx/x) = k$ (a constant), or $y = bx^k$ in which b is a constant, with the value y/x^k when y and x are an initial or final or other arbitrarily fixed size. y is the size of the part (e.g., an appendage), and x the size of the rest of the body (e.g., the trunk of the body). Where there is an inert mass (of size a) which does not grow, the formula: $y = (bx^k + a)$ may be found more suitable. In these formulæ the value k may be found by plotting the curve of growth on double logarithmic paper, which gives a straight line whose slope is k. This formula is useful for determining the relative rate of two parts or organs of the body.

Index Numbers

Collectively these constitute a device for indicating quantitative changes in economic (or other) conditions. The changes are expressed in percentages of a condition present in a fixed base year. Index numbers measure relative change. The numbers may be either simple or composite.

Simple index numbers show the changes in a single variable. Having selected a base year, divide the number or value for that year into the number or value for each of other years whose relation to the first is desired, and multiply the quotient by 100. Or each year may be taken in succession as a base year. In this case the index number indicates changes from one year to the next following. This latter method of computing index numbers is called the chain system.

Aggregative index numbers are either unweighted or weighted. The unweighted indices are obtained by summing the values for the group of 2 or more elements that are to be compared, first for the base year, taken as 100, then for some other year, and by dividing the sum for that year by the sum for the base year. This index is called aggregative. For example:

	Base year	5 years later	Aggregate index
Price of wool.......	0.5883	1.6600	
Price of mutton.....	0.1025	0.1982	
Price of skins.......	2.5833	5.5625	
Total..........	3.2741	7.4207	2.2665

Weighted indices are those in which prices of and number of related commodities are multiplied by some value-weight; it may be quantity of consumption (or production); it may be per cent of total consumption of each item in an aggregate. For example:

	Base year			Given year			
	Price	Consumption in millions of units	Product	Price	Consumption in millions of units base year	Product	Aggregative weighted index
Wool	0.5883	448	263.6	1.6600	448	743.7	
Mutton	0.1025	732	75.0	0.1982	732	145.1	
Skins	2.5833	6.7	17.3	5.5625	6.7	37.3	
			355.9				260.2

The weighted index is found by dividing the summed products of price for the given year and consumption for the base year by the summed price × consumption products for the base year.

Thus the aggregative weighted index shows that in the given year the products were 260 per cent of the base year.

Irving Fisher's (1922, p. 482) ideal formula for *weighted aggregative index* number is:

$$\sqrt{\left(\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}\right) \times \left(\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}\right)}$$

where p_1 stands for price at given year, p_0 price at base year; q_1 weight at given year, q_0 weight at base year.

Measuring Secular, Seasonal and Cyclical Change

1. *Rapid but Crude Methods*

For a fairly large number of recurring periods (months) find the total for each period (e.g., the total for all the Januaries; the total for all the Februaries, etc.). Find the average of the totals for all the months, and divide each monthly total by

this average. This gives the relative numbers for each month. To eliminate the seasonal factor divide each original item by this relative number. This crude method is inadequate if there is a marked trend, say downward, from January to December. This difficulty can be relieved by finding the total change from the January of the year under consideration to the succeeding January and subtracting from the item of the February total of the year one-twelfth of the total annual change; from the March total two-twelfths; from April three-twelfths, and so on. This adjustment for trends has to be made before the relative numbers are computed.

2. *Smoothing the Time Series When Trend Can be Expressed by a Straight Line*

In economics, and to a less extent in biology (animal or plant censuses and the like), the frequencies of production rates, etc., in successive periods of time often show great fluctuations (Table 11.) The problem is to replace the very irregular curve of time changes with a smooth trend line and to express the trend quantitatively.

The best measure of slope is the line of regression of which the formula is $\frac{\Sigma xY}{\Sigma x^2}$.

One arranges (as in Table 13) the X series (e.g., years) in the first column; the quantities (Y) found, in the second; and the deviations (x) of each of these elements of the time series (e.g., years) from the mean time in a third column; one then computes x^2, the squares of these deviations, for a fourth column. Next $x \cdot Y$ is calculated, and $\Sigma x \cdot Y$ is obtained, having due regard to sign. This sum is now divided by the sum of the x^2 column. Using this increment and starting from the assumed zero the smoothed production (trend) values are computed (Table 14). The quotient gives the increment in the unit of time (e.g., years). The line based on computed values (trend) may be passed through the mean point on the same graph paper with the irregular curve of time changes (Fig. 25).

The long-time trend may be eliminated by taking the regression line as a base and plotting the actual figures as plus or minus deviations therefrom. When this elimination is effected seasonal changes become still more striking (Fig. 26).

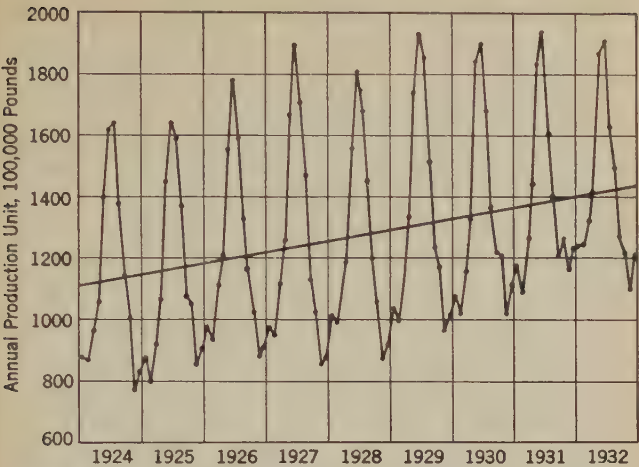

FIG. 25.—Production of creamery butter in factories of the United States, 1924–32. (From Yearbook of Department of Agriculture, 1934)

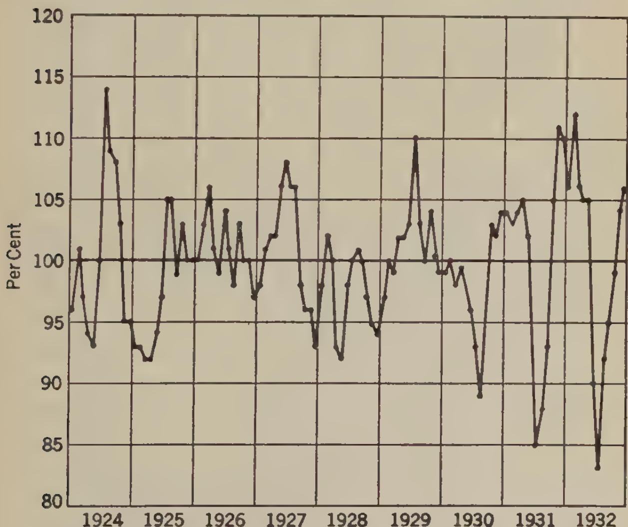

FIG. 26.—Creamery butter production, United States, 1924–32. Secular trend and seasonal variation eliminated

To take into account seasonal variation, find for each unit period (e.g., month) the frequency distribution of the ratio of actual to trend values (e.g., of production, Table 14), and then their arithmetic mean * for each period (Table 12). These means of quantities should then be adjusted so as to give an annual average of 100. These seasonal indices are then each subtracted from 100 to give the monthly deviations from the annual mean.

An illustration based on production of creamery butter is given in Table 12. The bottom line shows the seasonal variation.

Seasonal variation may be eliminated from any series by subtracting from or adding to such ratios the seasonal indices for each successive month. This method is illustrated for two years of production of creamery butter in Table 14.

Example. Using the data of Table 11 (p. 135), a table is made (Table 13). Column 1, the years (x series); column 2 the annual production of creamery butter in units of 100,000 pounds. Column 3, the deviations of the years from the middle year; column 4, the squares of the deviations; column 5, the product of the items in columns 2 and 3 $(x \cdot Y)$

$$\frac{\Sigma xY}{\Sigma x^2} = \frac{26573}{60} = 442.88, \text{ the computed increment per year.}$$

The computed middle year (1928) value is 15229.67 and $Y' = 15229.67 + 442.88x$, x being years. For July, 1928, the computed monthly production is $15230 \div 12 = 1269.17$, and the successive average changes in monthly production from a given month to the same month a year later is $1269.17 + 36.91x$, where x is the number of years. The monthly increase of monthly production is one-twelfth of the annual increase of monthly production, or 3.08. Thus on July 15, 1928, the computed monthly production is 1270.71, and on January 15, 1928, 1252.25.

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* If these ratios are erratically spaced the median should be used instead of the mean as a measure of central tendency.

TABLE 11.—CREAMERY BUTTER: PRODUCTION IN FACTORIES, UNITED STATES, 1923–1932.

Based on Table 398 of Yearbook of Agriculture (U.S.A.), 1934.

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Total
	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.	100000 lb.
1923	837	741	883	1005	1344	1584	1383	1208	1023	893	749	773	12422
1924	875	867	958	1060	1400	1620	1644	1378	1151	1005	773	830	13561
1925	871	802	923	1070	1455	1643	1589	1367	1083	1045	855	911	13615
1926	979	942	1124	1210	1559	1783	1596	1333	1167	1031	885	909	14518
1927	980	955	1115	1264	1688	1888	1705	1468	1135	1024	861	882	14965
1928	1010	994	1118	1188	1563	1810	1676	1454	1195	1059	877	925	14870
1929	1035	1000	1144	1337	1743	1929	1853	1522	1236	1181	972	1019	15970
1930	1084	1023	1157	1333	1844	1898	1676	1374	1226	1202	1020	1117	15952
1931	1184	1096	1268	1454	1838	1943	1613	1404	1209	1266	1170	1231	16675
1932	1243	1249	1331	1417	1866	1906	1634	1496	1274	1218	1098	1208	16941

TABLE 12.—PRODUCTION OF CREAMERY BUTTER: DISTRIBUTION OF ACTUAL TO TREND VALUES DURING 9 YEARS

Actual ÷ computed production of creamery butter in percents	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
150–159.9						1						
140–149.9						7	2					
130–139.9					6	1	4					
120–129.9					3		1	1				
110–119.9							2	5				
100–109.9				5				3	1			
90– 99.9			4	4					6	1		
80– 89.9	7	2	5						2	8	1	3
70– 79.9	2	7									5	6
60– 69.9											3	
Arithmetic mean	82.8	77.2	89.4	100.6	131.7	145.0	131.7	112.8	93.9	86.1	72.8	78.3
Seasonal indices (with base of 100)	82.6	77.1	89.2	100.4	131.5	144.7	131.5	112.6	93.7	85.9	72.7	78.2
100 − seasonal indices	17.4	22.9	10.8	−0.4	−31.5	−44.7	−31.5	−12.6	6.3	14.1	27.3	21.8

TABLE 13.—SHOWING HOW TREND LINE OF ANNUAL PRODUCTION OF CREAMERY BUTTER IN THE UNITED STATES IS COMPUTED

Year	Annual production (Y)	x	x^2	xY	Computed values
1932	16941	4	16	67764	17001
1931	16675	3	9	50025	16558
1930	15952	2	4	31904	16115
1929	15970	1	1	15970	15673
1928	14870	0	0	0	15230
1927	14965	−1	1	−14965	14787
1926	14518	−2	4	−29036	14344
1925	13615	−3	9	−40845	13901
1924	13561	−4	16	−54244	13458
	137067		60	26573	

Annual slope $= \dfrac{\Sigma xY}{\Sigma x^2} = 442.88$; average annual increase in total production.

$\dfrac{\Sigma Y}{n} = 15229.67$; middle year (1928) value.

$Y' = 15229.67 + 442.88x$ (x being the number of years).

For July, 1928, the computed (monthly) production is $\dfrac{15230}{12} = 1269.17$.

On a monthly basis the average changes of production from a given month to the same month a year later is

$$Y' = 1269.17 + 36.91x$$

where x is the number of years, and Y' is the computed value of Y.

Monthly increase in production is 3.08.

3. *Method of Moving Median (King)*

(1) Tabulate and plot the original series of data that is being studied at the monthly or other available interval. (2) Then draw a free-hand curve representing as approximately as possible the course of the cycle. Next, read from the free-hand curve the figures representing the tentative estimate of the cycle-amounts. Divide the actual data by the corresponding tentative estimate for each month (or other time unit). (3) These quotients are then tabulated as a table of ratios of actual (1) to tentative (2) cyclic data. (4) These are smoothed for each month by a moving median, which is then

TABLE 14.—TABLE OF PRODUCTION OF CREAMERY BUTTER IN THE UNITED STATES SMOOTHED AND ADJUSTED FOR LONG TREND AND SEASONAL VARIATION

Based on Table 398, Yearbook of Agriculture (U.S.A.), 1934

Year and month		Actual production in 100000 lb.	Computed production	Actual ÷ computed	Adjustment to be made for seasonal variation	Cyclical variation
1924	1	875	1104	79.3	17.4	96.7
	2	867	1107	78.3	22.9	101.2
	3	958	1111	86.2	10.8	97.0
	4	1060	1114	95.2	−.4	94.8
	5	1400	1117	125.3	−31.5	93.8
	6	1620	1120	144.6	−44.7	99.9
	7	1644	1123	146.4	−31.5	114.9
	8	1378	1126	122.4	−12.6	109.8
	9	1151	1129	101.9	6.3	108.2
	10	1005	1132	88.8	14.1	102.9
	11	773	1135	68.1	27.3	95.4
	12	830	1138	72.9	21.8	94.7
1925	1	871	1141	76.3	17.4	93.7
	2	802	1144	70.1	22.9	93.0
	3	923	1147	80.5	10.8	91.3
	4	1070	1151	93.0	−.4	92.6
	5	1455	1154	126.1	−31.5	94.6
	6	1643	1157	142.0	−44.7	97.3
	7	1589	1160	137.0	−31.5	105.5
	8	1367	1163	117.5	−12.6	104.9
	9	1083	1166	92.9	6.3	99.2
	10	1045	1169	89.4	14.1	103.5
	11	855	1172	73.0	27.3	100.3
	12	911	1175	77.5	21.8	99.3

assigned to the middle year spanned by this moving median as the smoothed value for the year and month. The median is usually to be preferred to the mean as being less influenced by large chance fluctuations. (5) Next adjust the percentages for each month of the year so that the sum of the adjusted monthly averages equals exactly 12, or in other cases equals exactly the number of repetitions of measurements for the year. Finally divide each item of the original raw tabulations (1) by the corresponding items in (5). The quotients give the cyclical movements after elimination of the normal seasonal movements.

The Median-Link relative method is a more refined method worked out by W. M. Persons (Rev. Economic Statistics, January, 1919). A good description of this method is given in Secrist, 1925, pp. 450–465.

Measure of Dissymmetry in Organisms

A dissymmetry index, Ξ, measuring the average degree of asymmetry in the right and left organs of bilateral organisms, was proposed by Duncker (1903).

First a series of integral differences $-3, -2, -1, 0, 1, 2, 3, 4$, etc., between the right- and left-side measurements of the organ in question is made, and the frequencies of each integral difference (reckoning to the nearest integer) is found. The average of the difference series is the difference of the averages of the right- and left-side measurements, and the standard deviation of the difference is given by

$$\sigma_d = \sqrt{\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 - 2r\sigma_{\mathrm{I}}\sigma_{\mathrm{II}}},$$

in which the subscripts refer to the bilateral series of which the asymmetry is to be found, and r is the coefficient of correlation between the two sides.

Let d' represent any positive differences in the series, and d'' any negative differences; and let f_1', f_2', etc., represent the frequencies of the negative-difference classes, and f_1'', f_2'', etc., the frequencies of the positive-difference classes. Then the asymmetry index

$$\Xi = \frac{\Sigma(f') \times \Sigma(d') - \Sigma(f'') \times \Sigma(d'')}{n[\Sigma(d') + \Sigma(d'')]} = 0.$$

Example. Absolute difference between dextral (d) and sinistral (s) lateral edges (L) of carapace of right-handed fiddler-crabs—*Gelasimus pugilator* (Yerkes, 1901; Duncker, 1903):

$d = L_d - L_s$:	-1	0	1	2	3
f:	1	63	310	23	3

$$\Sigma(d') = 310 \times 1 + 23 \times 2 + 3 \times 3 = 365, \qquad \Sigma(f') = 336.$$

$$\Sigma(d'') = 1, \quad \Sigma(f'') = 1, \quad n = 400.$$

$$\Xi = \frac{336 \times 365 - 1 \times 1}{400 \times 366} = \frac{122639}{146400} = 0.83770.$$

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EXPLANATION OF TABLES

I. Formulas. In this table the principal formulas used in the calculation of curves are brought together for convenient reference. The meanings of the letters are explained in the text. This table is preceded by an index to the principal letters used in the formulas of this book.

II. Certain constants and their logarithms. This table includes the constants most frequently employed in the calculations of this book.

III. Table of ordinates of normal curve. This table is for comparison of a normal frequency polygon consisting of weighted ordinates with the theoretical curve. The mode is taken at 10,000.

Example: $M = 17.673$; $\sigma = 1.117$; $y_0 = 181.4$.
(See p. 49.)

V	$V - M$	$\frac{V-M}{\sigma}$	Entries in Table corresponding to $\frac{V-M}{\sigma}$	y_0	y
14	−3.673	3.29	.00449	× 181.4 = 0.8	1
15	−2.673	2.39	.05750	× 181.4 = 10.4	8
16	−1.673	1.50	.32465	× 181.4 = 58.9	63

IIIa. Table of ordinates of normal curve, based on $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$ obtained from Table III by multiplying

each entry in that table by $\frac{1}{\sqrt{2\pi}} = 0.398942$ to get the respective entries in the Table. The mode is at $\frac{x}{\sigma} = 0.39894$.

IV. Table of values of probability integral. This table is for comparison of a normal frequency polygon consisting of rectangles with the theoretical curve.

Example: $AM = 17.673$; $\sigma = 1.1170$. (See p. 49.)

Class	$\frac{x}{\sigma}$	Per cent	Class Limits	Deviation from $A = x_1$	$\frac{x_1}{\sigma}$	$(\frac{1}{2} - \frac{1}{2}a) \times 100$ less $\sum_{x_1 \div \sigma}^{\infty}$
14	−3.29	0.2				0.225
			14.5	−3.173	−2.841	
15	−2.39	1.6				2.364
			15.5	−2.173	−1.945	
16	−1.50	12.4				12.097
			16.5	−1.173	−1.050	
17	−0.60	30.3				29.155
			17.5	−0.173	−0.155	
18	0.29	32.3				33.194
			18.5	0.827	0.740	
19	1.19	18.9				17.873
			19.5	1.827	1.636	
20	2.08	3.9				4.524
			20.5	2.827	2.531	
21	2.98	0.4				0.568
		100.0				100.000

In the example, the data of which are given on page 49, the frequency between the limits is given in per cent column. The $\frac{x}{\sigma}$ of each limit (as inner class limit) is found and the entries in Table IV corresponding to the limits are taken. Each such entry is subtracted from 0.50000, is multiplied by 100, and from the product is subtracted the total theoretical percentage of variates lying between the *outer* limit of the class and the corresponding extremity of the half curve. This gives the *theoretical* frequency of the class in question. The closeness of agreement of the last column with the "Per cent" column indicates the closeness of the observed frequency to the theoretical.

V. Table of reduction from the common to the metric system.

a. Inches to millimeters. This is first given for whole inches from 1 to 99, excepting even tens which may be got from the first line of figures by shifting the decimal point one place to the right. The table may be used for hundredths of an inch by shifting the decimal point two places to left. The reduction of twelfths and sixteenths of inches to millimeters is given immediately below.

b. The table for converting avoirdupois ounces into grams is based on the conversion factor given.

c. Table for converting pounds into grams, using the conversion factor given.

VI. Table of minutes and seconds of arc in decimals of a degree. This table will be found of use in the fitting of curves of Type IV (p. 61).

VII. First to sixth powers of integers from 1 to 30. This table is useful in calculating moments.

VIII. Factorials of integers. The product of all integers from 1 to and including the given integer, *i.e.*, $n!$ or $\underline{|n}$.

IX. Factors for use in obtaining the probable errors of the means and standard deviations, from $N = 5$ to $N = 250$. The entries of this table are computed from the fraction given at the top of the column, for the different values of N, given at the left of the entry. Since the probable error of the mean is $\frac{0.6745}{\sqrt{N}}\sigma$, and the probable error of the standard deviation is $\frac{0.6745}{\sqrt{2N}}\sigma$, the probable error in each case is obtained by multiplying the entry by σ. Standard errors are 1.483 times probable errors; roughly $\frac{1}{2}$ greater.

$$\text{Formula:}\quad PE_{mean} = \frac{0.67448975}{\sqrt{N}}\sigma;\quad PE\sigma = \frac{0.67448975}{\sqrt{2N}}\sigma.$$

Examples.

Let $N = 110$, mean $= 1542.50$, $\sigma = 60.45$.

$PE_{mean} = (0.064310)(60.45) = 3.89.$

Then mean $= 1542.50 \pm 3.89.$

$PE\sigma = (0.045474)(60.45) = 2.75.$

Then $\sigma = 60.45 \pm 2.75.$

The corresponding standard errors are approximately one-half greater than the probable errors or can be obtained to a greater degree of precision by dividing the probable errors by 0.6745.

Especially with smaller numbers of observations (N) use the entry for one less than N.

X. Odds against the occurrence of a deviation in terms of the standard deviation (of any individual in a distribution or a sample mean) as great or greater in a large collection of samples taken from a homogeneous population. This was computed from the formula:

$$\frac{1 - 2\left[1 - \frac{1}{2}(1 + \alpha)\right]}{2\left[1 - \frac{1}{2}(1 + \alpha)\right]}$$

in which α is double the area from the mode to x as given in Table IV. Thus in the example cited in explanation of Table XI, when $\frac{x}{\sigma} = 1.349$, the corresponding odds against are nearly midway between 4.17 and 5.19, or 4.68– as compared with 4.64 in the explanation of Table XI.

XI. Giving for deviations in multiples of the probable error from 1 to 5.0, in column 1; the probability of the occurrence of a deviation as great or greater in 100 trials, in column 2; also the odds against the occurrence of a deviation as great or greater in 100 trials $X : 1$ in column 3. Column 2 of this table is derivable from Table IV, in which the argument values of $\frac{x}{\sigma}$ are, of course, 0.67445 times those of $\frac{x}{PE}$, as given in this table. The corresponding entry is the integral for the half area of the probability curve. Double the entry, subtract from 1.00 and multiply by 100. The result is the entry of Table X. Thus, when $\frac{x}{PE}$ is 2.0, $\frac{x}{\sigma} = 1.349$. The nearest entry in Table IV is 0.41133. $0.41133 \times 2 = 0.82266$ and $100(1 - 0.82266) = 17.73$, the corresponding entry in Table X.

Column 3 gives the fraction that the probable non-occurrence (q, odds against) is of the probable occurrence (p), or $\frac{100 - p}{p} = \frac{q}{p}$. Thus, if 17.73 is the probable occurrence, $\frac{100 - 17.73}{17.73} = \frac{82.27}{17.73} = 4.64$. Thus the odds against are

4.64 to 1. From R. Pearl's "Medical Biometry and Statistics," W. B. Saunders Co.

XII. Values of χ^2. This table is copied from R. A. Fisher's "Statistical Methods for Research Workers" (Edinburgh: Oliver and Boyd) by generous permission of author and publishers. It gives for various values of $\chi^2 = \sum \frac{(f-y)^2}{y}$ (that is, the sum of all the differences between observed and expected frequency in each class expressed in units of the expected frequency) the probability that the found χ^2 may result only from errors of sampling. The χ^2 values are arranged in lines by number of degrees of freedom (n).

For example, if in $11 - 1$ classes one gets a χ^2 value of 10 then the probability of the deviation of the observed values, from expected being due to random sampling is about 0.40, a deviation which is to be expected in about 40 out of 100 samples, from the same source. But if χ^2 has the value 20, then the probability is only about 3 in 100, which is regarded as small. In fact, a probability (P) of less than 0.05 is regarded as so small as to justify the conclusion that the (large) observed differences between observation and expectation can not be due merely to random sampling.

XIII. Ratio (F) of the larger mean variance (as between means of samples and within samples) to the smaller mean variance. The ratios at the intersection of the column of the degrees of freedom of the greater mean variance and that of the smaller mean variance are the limiting ratios for an expectation that the ratio shall be exceeded in random sampling 5 times in 100 trials (plain type) or only once in 100 trials (bold-face type), hence a ratio that is very significant of a real difference between the sample such as can hardly be due to random sampling. Note that Fisher's z can be obtained from the F ratio by taking the square root of $\frac{s_1^2}{s_2^2}$ and finding $\log_e$ of this root. (From G. W. Snedecor, 1934, by kind permission of the author and publisher and with extensions by the author.)

Also in the right-hand column the values of t; t = difference between two means divided by the standard error of the difference (p. 38). In general, t measures the probability that

two samples belong to the same universe; t has to be considered in relation to the degrees of freedom in both of the samples. The entries in the right-hand column give the 5 per cent probability and the 1 per cent probability (in bold-face type) that in view of the t found the difference is significant of a real difference and not merely one due to sampling errors. If the t found is smaller than either of these numbers the difference is probably due merely to sampling. (This table also is from Snedecor, 1934.)

XIV. Table of the probable errors of the coefficient of correlation for various numbers of observations or variates (N) and for various values of r. The probable error of the coefficient of correlation being $\frac{0.6745(1 - r^2)}{\sqrt{N}}$, a table for the varying values of N and r is easily constructed, and for large values of N is accurate with interpolation by inspection to two significant figures, which are all that are required.

XV. Values of $1 - r^2$ for the various values of r from 0.00 to 0.99. This table will facilitate the computation of the probable error of r, in accordance with the formula: $PE_r = 0.67449 \times \frac{1 - r^2}{\sqrt{N}}$; or the standard error of r, $\frac{1 - r^2}{\sqrt{N}}$. This table has been specially computed.

XVI. Values of $\sqrt{1 - r^2}$, corresponding to values of r from 0.00 to 0.99. This table will be found of use in computing partial correlations and that of the average variability in an array in the correlation tables, whose value is $\sigma\sqrt{1 - r^2}$ and in many other cases. Specially computed. Tables of values of $1 - r^2$ and $\sqrt{1 - r^2}$ have been published by Miner (1922), and copied by others.

XVII. Values of $r = 2 \sin \frac{\pi}{6} \rho$ for different values of ρ. This will aid in computing r by the rank difference method, given at page 79.

XVIII. Functions of r_t for use in computing the approximate probable error of the tetrachoric coefficient of correlation function;

$$0.6745 \sqrt{[1 - r_t^2]\left[1 - \left(\frac{\sin^{-1} r_t}{90°}\right)^2\right]}$$

where r_t is the tetrachoric coefficient. The entire formula (p. 108) is:

$$PE_{r_t} = \left\{0.6745 \sqrt{[1 - r_t^2]\left[1 - \left(\frac{\sin^{-1} r_t}{90^\circ}\right)^2\right]}\right\}$$

$$\left\{\frac{\sqrt{(a+b)(a+c)(c+d)(b+d)}}{N^2 yy' \sqrt{N}}\right\}$$

in which y, y' are the ordinate values of the normal probability curve, obtained from Table IIIa; see page 165.

XIX. Table of log Γ functions of p. This table will enable one to solve the equations for y_0 given on pages 55–57 The table gives the logarithms of the values of Γ functions only within the range $p = 1$ to 2. As all values of the function within these limits are less than 1, the mantissa of the logarithms is -1; but it is given in the table as $10 - 1 = 9$, as is usually done in logarithmic tables.

Supposing the quantity of which we wish to find the value reduced to the form $\Gamma(4.273)$. The value can not be found directly because the value of p is larger than the numbers in the table (1 to 2). The solution is made by aid of the equation $\Gamma(p + 1) = p\Gamma(p)$, thus:

$$\begin{aligned}
\log \Gamma(1.273) &= 9.955185 \\
\log \quad 1.273 &= 0.104828 \\
\hline
\log \Gamma(2.273) &= 0.060013 \\
\log \quad 2.273 &= 0.356599 \\
\hline
\log \Gamma(3.273) &= 0.416612 \\
\log \quad 3.273 &= 0.514946 \\
\hline
\log \Gamma(4.273) &= 0.931558
\end{aligned}$$

or, more briefly,

$$\begin{aligned}
\log \Gamma(1.273) &= 9.955185 \\
\log \quad 1.273 &= .104828 \\
\log \quad 2.273 &= .356599 \\
\log \quad 3.273 &= .514946 \\
\hline
\log \Gamma(4.273) &= 0.931558 = \log 8.542
\end{aligned}$$

XX. Theoretical value of Q for various sizes of families (s) of 2 to 17 children. Q is the proportion that V is of $U + V$ or $Q = \frac{V}{U+V}$. To get U and V one has first to know the genic constitution of each of the two traits whose linkage is being studied as indicated in the text (p. 126).

XXI. Standard errors of the mean theoretical values of Q given in Table XX. To use in judging to which crossing-over category of Table XX any determination of Q belongs.

XXII. Giving for any unit character the proportion of recessive offspring to be expected: (R) in random matings of dominants with dominants, and (S) in random matings of dominants with recessives when (B) gives the proportion of recessive individuals in the general population. The formulæ are:

$$R = \frac{1}{4}\left(\frac{2pq}{q^2 + 2pq}\right)^2 = \left(\frac{p}{q + 2p}\right)^2$$

$$S = \frac{1}{2}\left(\frac{2pq}{q^2 + 2pq}\right) = \frac{p}{q + 2p}$$

where $p + q = 1$.

$q^2 + 2pq =$ the dominant individuals in the population (A).

$p^2 =$ the recessive individuals in the population (B).

$p = \sqrt{b}$.

$q = 1 - \sqrt{b}$.

The probable error of $\frac{p}{q + 2p}$ is given by the formula:

$$\frac{0.6745}{8N(1 + \sqrt{b})}$$

$$\sqrt{\frac{16N(1-\sqrt{b})}{1+\sqrt{b}} - \frac{(1-\sqrt{b})^2}{b} - \frac{4(1-\sqrt{b})^2}{(1+\sqrt{b})^2\sqrt{b}} - \frac{4(1-\sqrt{b})^2}{(1+\sqrt{b})^2}}.$$

When N is 100 or more this formula may be simplified as follows:

$$PE \text{ of } \frac{p}{q+2p} = \frac{0.6745(1 - \sqrt{b})}{2(1 + \sqrt{b})}\sqrt{\frac{1}{(1-b)N}}.$$

The probable error of $\left(\frac{p}{q+2p}\right)^2$ is given as

$$\frac{0.6745(1-\sqrt{b})}{4N(1+\sqrt{b})^2}\sqrt{\frac{16Nb}{1-b}-1-\frac{(1-2\sqrt{b})^2}{(1+\sqrt{b})^2}+\frac{2(1-2\sqrt{b})}{1+\sqrt{b}}}.$$

When N is 100 or more this reduces to approximately:

$$PE \text{ of } \left(\frac{p}{q+2p}\right)^2 = \frac{0.6745(1-\sqrt{b})}{(1+\sqrt{b})^2}\sqrt{\frac{b}{N(1-b)}}.$$

XXIII. Squares, cubes, square roots, and reciprocals of numbers from 1 to 1054. The use of this table can be extended by using the principle that, if any number be multiplied by n, its square is multiplied by n^2, its cube by n^3 and its reciprocal by $\frac{1}{n}$.

INDEX TO PRINCIPAL LETTERS USED IN THE FORMULÆ OF THIS BOOK

- AD Average deviation
- a Frequency of upper left quadrant
- α Skewness index
- b The frequency of the upper right quadrant (p. 105)
- β Ratio of moments (p. 43)
- CC Coefficient of contingency
- CC' Corrected coefficient of contingency
- c The frequency of the lower left quadrant
- D Distance from mean to mode (pp. 53–58); also stands for dominant
- d A difference; differential; the frequency of the lower left quadrant
- δ_1 Difference between y and f; other deviations
- e Base of Napierian logarithms = 2.718282
- η Correlation ratio
- F Critical function (pp. 44–49); ratio of variances (pp 38, 70)
- f Class frequency
- h A certain value of x (p. 39)
- i Class interval or range

$I_{p'p''}$	Interval between the p'th and p''th individual (p. 51)
i_p	Interval between the pth and $(p + 1)$th individual (p. 50)
K	A function of k
k	A proportional value of x; also any constant; also coefficient of alienation (p. 91)
κ	Factor in finding limiting deviation from M (p. 21)
l	Range of normal curve along the abscissal axis (p. 53)
$l_1 l_2$	Portions of the curve range (p. 54)
Λ	Number of classes
M	Arithmetic mean
m	Modulus of common system of logs = log e
M'	Arbitrary origin, assumed mean
M_{a+b}	Mean of composite
M_G	Geometric mean
M_H	Harmonic mean
M_0	Abscissal value of the mode (theoretical)
M'_0	Abscissal value of the mode (empirical)
M_s	Arithmetic mean of sample
μ	Moment about M
N	Number of variates (p. 24).
n	Number of classes
$n!\ \lfloor n$	Factorial n, product of all integers from 1 to n (p. 150)
$\mathfrak{N}$	Number corresponding to logarithm: antilog
ν_1	Average moment about M'
P	Probability
p	Probability that an event will occur, especially the occurrence of a recessive trait or gene; ordinal rank of a particular individual or case (p. 50)
PE	Probable error of the determination of any value
PE_{AD}	Probable error of the average deviation

Q (In economics), quantity of production; Q_0 in base year; Q_1 in current year. Also quartile deviation

R Recessive; coefficient of multiple correlation

r Coefficient of correlation

ρ Coefficient of correlation by rank-difference method, p. 78

s A relation of β's (p. 44); also "sample"; also standard deviation of sample

SE Standard error of the distribution of the magnitudes

SE_M Standard error of the mean, $= \sigma/\sqrt{N}$ = standard deviation of the sampling distribution of the mean

SE_σ Standard error of a standard deviation, $\sigma/\sqrt{2N}$

Σ Summation sign

σ Standard deviation of a frequency distribution; index of variability

t Measure of the significance of a difference (p. 38, 76); also number of observed recessives

τ In type IV (p. 56); number of expected recessives

θ, ϕ Angles

U Sum in mean square contingency formula; also number of non-crossover progeny

V Abscissal value of any class; coefficient of variation

V_0 Value of central class

v Magnitude of any variate or value

X The horizontal axis or basis of polygon

x A varying abscissal value

x_1, x_2, etc. Definite values of x

χ Test of closeness of fit (p. 47)

Y The vertical axis of polygons; also the log of y (p. 52)

y A varying ordinate value

y_0 Value of the ordinate at the origin

z' Transformed r (p. 76)

TABLE I. FORMULÆ

Binomial expansion

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + y^n$$

To solve any equation of the second degree:

$$ax^2 + bx + c = 0; \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Average $\qquad M = \dfrac{\Sigma v}{N} = \dfrac{\Sigma fV}{N}$

Geometric mean

$$G = \sqrt[N]{v_1 \cdot v_2 \cdot v_3 \cdot v_4 \ldots} \text{ or } \log G = \frac{\Sigma(\log v)}{N}$$

$$\text{Standard deviation} = \sigma = \sqrt{\frac{\Sigma(v-M)^2}{N}}$$

$$= \sqrt{\frac{\Sigma f(V-M)^2}{N}} = i\sqrt{\frac{\Sigma f(V-V_0)^2}{N} - \left[\frac{\Sigma f(V-V_0)}{N}\right]^2}$$

Variance $= \sigma^2$ $\qquad$ Coefficient of variation $C = \dfrac{\sigma}{M} \times 100$

Probable and Standard Errors

$$PE_M = \pm\frac{0.6745\sigma}{\sqrt{N}} \qquad PE_\sigma = \pm\frac{0.6745\sigma}{\sqrt{2N}}$$

$$SE_M = \mp\frac{\sigma}{\sqrt{N}} \qquad SE_\sigma = \mp\frac{\sigma}{\sqrt{2N}}$$

$$PE_{Mdn} = \pm\frac{0.8454\sigma}{\sqrt{N}} \qquad PE_C = \pm\frac{0.6745C}{\sqrt{2N}}\left[1 + 2\left(\frac{C}{100}\right)^2\right]^{1/2}$$

Significant difference $> 3\sqrt{PE_{Mx}{}^2 + PE_{My}{}^2}$, or $2\sqrt{SE_{Mx}{}^2 + SE_{My}{}^2}$

TABLE I. FORMULÆ—*Continued*

Significance of difference between correlation coefficients of 2 samples,

$$z' = \tfrac{1}{2}\log_e(1 + r) - \log_e(1 - r), \text{ p. } 76$$

Standard error of z', $\mp \dfrac{1}{\sqrt{n' - 3}}$

Curve Analysis

$$\nu_1 = \frac{\Sigma f(V - V_0)}{N} \qquad \nu_2 = \Sigma f(V - V_0)^2$$

$$\nu_3 = \frac{\Sigma f(V - V_0)^3}{N} \qquad \nu_4 = \frac{\Sigma f(V - V_0)^4}{N}$$

$$\mu_1 = 0 \qquad \mu_2 = \nu_2 - \nu_1^2 \qquad \mu_3 = \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3$$

$$\mu_4 = \nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4$$

$$\mu_1' = 0 \quad \mu_2' = \nu_2 - \nu_1^2 - \tfrac{1}{12}; \quad \mu_3' = \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3$$

$$\mu'_4 = \nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4 - \tfrac{1}{2}(\nu_2 - \nu_1^2) + \tfrac{7}{240}$$

For normal distribution:

$$SE\mu_2 = \mp\,\sigma^2\sqrt{\frac{2}{N}} \qquad SE\beta_1 = \sqrt{\frac{6}{N}}$$

$$SE\mu_3 = \mp\,\sigma^3\sqrt{\frac{6}{N}} \qquad SE\beta_2 = \sqrt{\frac{24}{N}}$$

$$SE\mu_4 = \mp\,\sigma^4\sqrt{\frac{96}{N}}$$

Chi square, $\chi^2 = \sum \dfrac{\delta_1^2}{y}$

For skew distributions:

Distance mean to mode: $D = \sigma \cdot \alpha$

Skewness $\alpha = \frac{1}{2}\sqrt{\beta_1}\,\dfrac{s \pm 2}{s \mp 2}$ (Types I, IV); $\alpha = \dfrac{2\sqrt{p - 3}}{p}$ (Type V)

TABLE I. FORMULÆ—*Continued*

$$SE_D = \mp\sqrt{\frac{3}{2N}}\,\sigma$$

Curve Typing

$$F_1 = 2\beta_2 - 3\beta_1 - 6$$

$$F = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)\,(2\beta_2 - 3\beta_1 - 6)}$$

Variance, Equal Classes

Total sum of squares, $\Sigma(x - Mx)^2$

Sum of squares within classes, $\Sigma_s\Sigma(x - M_s)^2$

Sum of squares between classes, $n\Sigma_s(M_s - M_x)^2$

Test Significance of a Difference in Variance

$$F = \frac{\text{larger mean square}}{\text{smaller mean square}} \text{ (with Table XIII)}$$

Correlations, etc.

$$r = \frac{\Sigma(\text{dev } x \times \text{dev } y \times f)}{N\cdot\sigma_1\cdot\sigma_2}$$

$$SE_r = \mp\frac{1 - r^2}{\sqrt{N}} \text{ or, small samples: } \frac{1 - r^2}{\sqrt{N - 1}}$$

Grouped data

$$r_{xy} = \frac{\Sigma fx'\cdot y' - \dfrac{(\Sigma fx')\,(\Sigma fy')}{N}}{\sqrt{\Sigma f(x')^2 - \dfrac{(\Sigma fx')^2}{N}}\sqrt{\Sigma f(y')^2 - \dfrac{(\Sigma fy')^2}{N}}}$$

Regression equation: To predict Y from X:

$$Y = a + bX;\quad a = My - bMx;\quad b = \frac{\sigma_y}{\sigma_x}r$$

To predict X from Y:

$$X = a + bY;\quad a = Mx - bMy;\quad b = \frac{\sigma_x}{\sigma_y}r$$

TABLE I. FORMULÆ—*Continued*

Correlation ratio (η): $\eta_{yx} = \dfrac{\sigma_{M_y}}{\sigma_y}$; $\eta_{xy} = \dfrac{\sigma_{M_x}}{\sigma_x}$

Coefficient of alienation, $k = \sqrt{1 - r^2}$

Coefficient of determination, r^2 (p. 92)

Coefficient of reliability, $r_{(z_1+z'_1)(z_I+z'_I)} = \dfrac{2r_{1_I}}{1 + r_{1_I}}$ (p. 92)

Spurious correlation, $r_0 = \dfrac{V_c^2}{\sqrt{V_a^2 + V_c^2}\sqrt{V_b^2 + V_c^2}}$

Partial correlation, $r_{12.3} = \dfrac{r_{12} - r_{13} \cdot r_{23}}{\sqrt{1 - r^2_{13}}\sqrt{1 - r^2_{23}}}$ (p. 109)

Partial sigmas, $\sigma_{1.234\ldots n}$

$= \sigma_1 \sqrt{1 - r^2_{12}}\sqrt{1 - r^2_{13.2}}\sqrt{1 - r^2_{14.23}} \cdots \sqrt{1 - r^2_{1n.23.(n-1)}}$ (p. 112)

Multiple correlation $R_{1.234\ldots n}$

$= \sqrt{1 - [(1 - r^2_{12})(1 - r^2_{13.2})(1 - r^2_{14.23}) \ldots (1 - r^2_{1n.23(n-1)})]}$ (p. 116)

Coefficient of part correlation,

$${}_{12}r_{34} = \sqrt{\frac{b^2_{12.34}\sigma_2^2}{b^2_{12.34}\sigma_2^2 + \sigma_1^2(1 - R^2_{1.234})}} \quad \text{(p. 116)}$$

where $b_{12.34} = r_{12.34}\dfrac{\sigma_{1.234}}{\sigma_{2.134}}$ (p. 117)

Law of relative growth, $y = bx^k + a$ (p. 130)

Measure of slope in time series, $\dfrac{\Sigma xY}{\Sigma x^2}$ (p. 132)

II.—CERTAIN CONSTANTS AND THEIR LOGARITHMS

Title	Symbol	Number	Log
Ratio of circumference to diameter	π	3.1415927	0.4971499
Reciprocal of same	$\frac{1}{\pi}$	0.3183099	9.5028501
Square root of same	$\sqrt{\pi}$	1.7724538	0.2485749
Reciprocal of square root of same	$\frac{1}{\sqrt{\pi}}$	0.5641896	9.7514251
Square root of 2π	$\sqrt{2\pi}$	2.506628	0.399090
Reciprocal of same	$\frac{1}{\sqrt{2\pi}}$	0.3989423	9.6009101
Reciprocal of 2π	$\frac{1}{2\pi}$	0.159155	9.201820
Square root of 2	$\sqrt{2}$	1.4142136	0.150515
Reciprocal of same	$\frac{1}{\sqrt{2}}$	0.707107	9.8494849
Square root of $\frac{2}{\pi}$	$\sqrt{\frac{2}{\pi}}$	0.797885	9.9019401
Base of hyperbolic logarithms	e	2.7182818	0.4342945
Reciprocal of square root of same	$\frac{1}{\sqrt{e}}$	0.606530	9.7828528
Modulus of common system of logs = log e	m	0.4342945	9.6377843
Reciprocal of same = hyp. log 10	$\frac{1}{m}$	2.3025851	0.3622157
Factor to reduce σ to probable error	T	0.67449	9.828976
Com. log $x = m \times$ hyp. log x, or			
Com. log (com. log x) $= 9.6377843 +$ com. log (hyp. log x)			
Hyp. log $x =$ com. log $x \times \frac{1}{m}$, or			
Com. log (hyp. log x) = com. log (com. log) $x + 0.3622157$			
Circumference of circle	$2\pi r$		
Area of circle	πr^2		
Area of sector (length of arc $= l$)	$\frac{1}{2} lr$		
Area of sector (angle of arc $= a°$)	$\frac{a}{360}\pi r^2$		

Eccentricity of an ellipse, $\epsilon = \frac{\sqrt{a^2 - b^2}}{a^2}$, where a = semi-major axis; b = semi-minor axis of ellipse.

TABLE III.—TABLE OF ORDINATES (Y) OF NORMAL CURVE, OR VALUES OF $\frac{y}{y_0}$ CORRESPONDING TO VALUES OF $\frac{x}{\sigma}$

x = deviation from mean. y = frequency.

σ = standard deviation. $y_0 = \frac{N}{\sigma\sqrt{2\pi}}$ = maximum frequency.

x/σ	0	1	2	3	4	5	6	7	8	9
0.0	100000	99995	99980	99955	99920	99875	99820	99755	99685	99596
0.1	99501	99396	99283	99158	99025	98881	98728	98565	98393	98211
0.2	98020	97819	97609	97390	97161	96923	96676	96420	96156	95882
0.3	95600	95309	95010	94702	94387	94055	93723	93382	93034	92677
0.4	92312	91939	91558	91169	90774	90371	89961	89543	89119	88688
0.5	88250	87805	87353	86896	86432	85962	85488	85006	84519	84060
0.6	83527	83023	82514	82010	81481	80957	80429	79896	79359	78817
0.7	78270	77721	77167	76610	76048	75484	74916	74342	73769	73193
0.8	72615	72033	71448	70861	70272	69681	69087	68493	67896	67298
0.9	66698	66097	65494	64891	64287	63683	63077	62472	61865	61259
1.0	60653	60047	59440	58834	58228	57623	57017	56414	55810	55209
1.1	54607	54007	53409	52812	52214	51620	51027	50437	49848	49260
1.2	48675	48092	47511	46933	46357	45783	45212	44644	44078	43516
1.3	42956	42399	41845	41294	40747	40202	39661	39123	38589	38058
1.4	37531	37007	36487	35971	35459	34950	34445	33944	33447	32954
1.5	32465	31980	31500	31023	30550	30082	29618	29158	28702	28251
1.6	27804	27361	26923	26489	26059	25634	25213	24797	24385	23978
1.7	23575	23176	22782	22392	22008	21627	21251	20879	20511	20148
1.8	19790	19436	19086	18741	18400	18064	17732	17404	17081	16762
1.9	16448	16137	15831	15530	15232	14939	14650	14364	14083	13806
2.0	13534	13265	13000	12740	12483	12230	11981	11737	11496	11259
2.1	11025	10795	10570	10347	10129	09914	09702	09495	09290	09090
2.2	08892	08698	08507	08320	08136	07956	07778	07604	07433	07265
2.3	07100	06939	06780	06624	06471	06321	06174	06029	05888	05750
2.4	05614	05481	05350	05222	05096	04973	04852	04734	04618	04505
2.5	04394	04285	04179	04074	03972	03873	03775	03680	03586	03494
2.6	03405	03317	03232	03148	03066	02986	02908	02831	02757	02684
2.7	02612	02542	02474	02408	02343	02280	02218	02157	02098	02040
2.8	01984	01929	01876	01823	01772	01723	01674	01627	01581	01536
2.9	01492	01449	01408	01367	01328	01289	01252	01215	01179	01145
3	01111	00819	00598	00432	00309	00219	00153	00106	00073	00050
4	00034	00022	00015	00010	00006	00004	00003	00002	00001	00001
5	00000									

TABLE IIIa.—ORDINATES BASED ON $y = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$ OBTAINED FROM TABLE III BY MULTIPLYING EACH ENTRY IN THAT TABLE BY $\frac{1}{\sqrt{2\pi}} = 0.398942$ TO GET RESPECTIVE ENTRIES IN THIS TABLE.

(Specially calculated.)

x/σ	0	1	2	3	4	5	6	7	8	9
0.0	.39894	.39892	.39886	.39876	.39862	.39844	.39822	.39797	.39767	.39733
0.1	.39695	.39654	.39608	.39558	.39505	.39448	.39387	.39322	.39253	.39181
0.2	.39104	.39024	.38940	.38853	.38762	.38667	.38568	.38466	.38361	.38251
0.3	.38139	.38023	.37903	.37780	.37654	.37524	.37391	.37255	.37115	.36973
0.4	.36827	.36678	.36526	.36371	.36213	.36053	.35889	.35722	.35553	.35381
0.5	.35206	.35029	.34849	.34667	.34482	.34294	.34104	.33912	.33718	.33521
0.6	.33322	.33121	.32918	.32713	.32506	.32297	.32086	.31874	.31659	.31443
0.7	.31225	.31006	.30785	.30563	.30339	.30114	.29887	.29659	.29430	.29200
0.8	.28969	.28737	.28504	.28269	.28034	.27798	.27562	.27324	.27086	.26848
0.9	.26608	.26369	.26129	.25888	.25647	.25406	.25164	.24923	.24681	.24439
1.0	.24197	.23955	.23713	.23471	.23230	.22988	.22747	.22506	.22265	.22025
1.1	.21785	.21546	.21307	.21068	.20831	.20594	.20357	.20121	.19886	.19652
1.2	.19419	.19186	.18954	.18724	.18494	.18265	.18037	.17810	.17585	.17360
1.3	.17137	.16915	.16694	.16474	.16256	.16038	.15822	.15608	.15395	.15183
1.4	.14973	.14764	.14556	.14350	.14146	.13943	.13742	.13542	.13344	.13147
1.5	.12952	.12758	.12566	.12376	.12188	.12001	.11816	.11632	.11450	.11270
1.6	.11092	.10915	.10741	.10567	.10396	.10226	.10059	.09892	.09728	.09566
1.7	.09405	.09246	.09089	.08933	.08780	.08628	.08478	.08329	.08183	.08038
1.8	.07895	.07754	.07614	.07477	.07341	.07206	.07074	.06943	.06814	.06687
1.9	.06562	.06438	.06316	.06195	.06076	.05959	.05844	.05730	.05618	.05508
2.0	.05399	.05292	.05186	.05082	.04980	.04879	.04780	.04682	.04586	.04491
2.1	.04398	.04307	.04217	.04128	.04041	.03955	.03871	.03788	.03706	.03626
2.2	.03547	.03470	.03394	.03319	.03246	.03174	.03103	.03034	.02965	.02898
2.3	.02833	.02768	.02705	.02643	.02582	.02522	.02463	.02406	.02349	.02294
2.4	.02239	.02186	.02134	.02083	.02033	.01984	.01936	.01888	.01842	.01797
2.5	.01753	.01709	.01667	.01625	.01585	.01545	.01506	.01468	.01430	.01394
2.6	.01358	.01323	.01289	.01256	.01223	.01191	.01160	.01130	.01100	.01070
2.7	.01042	.01014	.00987	.00960	.00935	.00909	.00885	.00860	.00837	.00814
2.8	.00792	.00770	.00748	.00727	.00707	.00687	.00668	.00649	.00631	.00613
2.9	.00595	.00578	.00562	.00545	.00530	.00514	.00499	.00485	.00470	.00457
3	.00443	.00327	.00238	.00172	.00123	.00087	.00061	.00042	.00029	.00020
4	.00013	.00009	.00006	.00004	.00002	.00002	.00001	.00001	.00000	.00000

TABLE IV.—TABLE OF THE HALF CLASS INDEX ($\frac{1}{2}a$) VALUES OR THE NORMAL PROBABILITY INTEGRAL CORRESPONDING TO VALUES OF $\frac{x}{\sigma}$; OR THE FRACTION OF THE AREA OF THE CURVE BETWEEN THE LIMITS 0 AND $+\frac{x}{\sigma}$, OR 0 AND $-\frac{x}{\sigma}$

Total area of curve assumed to be 100,000

x = deviation from mean

σ = standard deviation

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
0.00	00000	40	80	120	159	199	239	279	319	359	40
0.01	0399	439	479	519	559	598	638	678	718	758	
0.02	0798	838	878	917	957	997	1037	1077	1117	1157	
0.03	1197	1237	1276	1316	1356	1396	1436	1476	1516	1555	
0.04	1595	1635	1675	1715	1755	1795	1834	1874	1914	1954	
0.05	1994	2034	2074	2113	2153	2193	2233	2273	2313	2352	
0.06	2392	2432	2472	2512	2551	2591	2631	2671	2711	2751	
0.07	2790	2830	2870	2910	2949	2989	3029	3069	3109	3148	
0.08	3188	3228	3268	3307	3347	3387	3427	3466	3506	3546	
0.09	3586	3625	3665	3705	3744	3784	3824	3864	3903	3943	
0.10	3983	4022	4062	4102	4141	4181	4221	4261	4300	4340	
0.11	4380	4419	4459	4498	4538	4578	4617	4657	4697	4736	
0.12	4776	4815	4855	4895	4934	4974	5013	5053	5093	5132	
0.13	5172	5211	5251	5290	5330	5369	5409	5448	5488	5527	
0.14	5567	5606	5646	5685	5725	5764	5804	5843	5883	5922	
0.15	5962	6001	6041	6080	6119	6159	6198	6238	6277	6317	
0.16	6356	6395	6435	6474	6513	6553	6592	6631	6671	6710	
0.17	6750	6789	6828	6867	6907	6946	6985	7025	7064	7103	
0.18	7142	7182	7221	7260	7299	7338	7378	7417	7456	7495	
0.19	7535	7574	7613	7652	7691	7730	7769	7809	7848	7887	
0.20	7926	7965	8004	8043	8082	8121	8160	8199	8238	8278	
0.21	8317	8356	8395	8434	8473	8512	8551	8590	8628	8667	39
0.22	8706	8745	8784	8823	8862	8901	8940	8979	9018	9057	
0.23	9095	9134	9173	9212	9250	9289	9328	9367	9406	9445	
0.24	9483	9522	9561	9600	9638	9677	9716	9754	9793	9832	
0.25	9871	9909	9948	9986	10025	10064	10102	10141	10180	10218	
0.26	10257	10295	10334	10372	10411	10449	10488	10526	10565	10603	
0.27	10642	10680	10719	10757	10796	10834	10872	10911	10949	10988	
0.28	11026	11064	11103	11141	11179	11217	11256	11294	11333	11371	
0.29	11409	11447	11485	11524	11562	11600	11638	11676	11715	11753	
0.30	11791	11829	11867	11905	11943	11981	12019	12058	12096	12134	
0.31	12172	12210	12248	12286	12324	12362	12400	12438	12476	12514	38
0.32	12552	12589	12627	12665	12703	12741	12778	12816	12854	12892	
0.33	12930	12968	13005	13043	13081	13118	13156	13194	13232	13269	
0.34	13307	13344	13382	13420	13457	13495	13533	13570	13608	13645	
0.35	13683	13720	13758	13795	13833	13870	13908	13945	13983	14020	

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1
38	3.8	7.6	11.4	15.2	19.0	22.8	26.6	30.4	34.2
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3

TABLE IV.—*Continued*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
0.36	14058	14095	14132	14169	14207	14244	14281	14319	14356	14393	
0.37	14431	14468	14505	14542	14579	14617	14654	14691	14728	14765	
0.38	14803	14840	14877	14914	14951	14988	15025	15062	15099	15136	37
0.39	15173	15210	15247	15284	15321	15357	15394	15431	15468	15505	
0.40	15542	15579	15616	15652	15689	15726	15763	15799	15836	15873	
0.41	15910	15946	15983	16019	16056	16093	16129	16166	16202	16239	
0.42	16276	16312	16348	16385	16421	16458	16494	16531	16567	16604	
0.43	16640	16676	16713	16749	16785	16821	16858	16894	16930	16967	
0.44	17003	17039	17075	17111	17147	17184	17220	17256	17292	17328	
0.45	17364	17400	17436	17472	17508	17544	17580	17616	17652	17688	36
0.46	17724	17760	17796	17831	17867	17903	17939	17975	18011	18046	
0.47	18082	18118	18153	18189	18225	18260	18296	18332	18367	18403	
0.48	18439	18474	18509	18545	18580	18616	18651	18687	18722	18758	
0.49	18793	18829	18864	18899	18934	18969	19005	19040	19075	19111	
0.50	19146	19181	19216	19251	19287	19322	19357	19392	19427	19462	
0.51	19497	19532	19567	19602	19637	19672	19707	19742	19777	19812	35
0.52	19847	19881	19916	19951	19986	20020	20055	20090	20125	20160	
0.53	20194	20229	20263	20298	20332	20367	20402	20436	20471	20505	
0.54	20540	20574	20609	20643	20678	20712	20746	20781	20815	20850	
0.55	20884	20918	20952	20986	21021	21055	21089	21123	21158	21192	
0.56	21226	21260	21294	21328	21362	21396	21430	21464	21498	21532	34
0.57	21566	21600	21634	21667	21701	21735	21769	21803	21836	21870	
0.58	21904	21938	21971	22005	22039	22072	22106	22139	22173	22207	
0.59	22240	22274	22307	22341	22374	22407	22441	22474	22508	22541	
0.60	22575	22608	22641	22674	22707	22741	22774	22807	22840	22874	
0.61	22907	22940	22973	23006	23039	23072	23105	23138	23171	23204	33
0.62	23237	23270	23303	23335	23368	23401	23434	23467	23499	23532	
0.63	23565	23598	23630	23663	23695	23728	23761	23793	23826	23859	
0.64	23891	23924	23956	23988	24021	24053	24085	24118	24150	24183	
0.65	24215	24247	24280	24312	24344	24376	24408	24441	24473	24505	
0.66	24537	24569	24601	24633	24665	24697	24729	24761	24793	24825	32
0.67	24857	24889	24920	24952	24984	25016	25048	25079	25111	25143	
0.68	25175	25206	25238	25269	25301	25332	25364	25395	25427	25459	
0.69	25490	25521	25553	25584	25615	25647	25678	25709	25741	25772	
0.70	25804	25835	25866	25897	25928	25959	25990	26021	26052	26084	
0.71	26115	26146	26176	26207	26238	26269	26300	26331	26362	26393	31
0.72	26424	26454	26485	26516	26546	26577	26608	26638	26669	26700	
0.73	26730	26761	26791	26822	26852	26883	26913	26943	26974	27004	
0.74	27035	27065	27095	27125	27156	27186	27216	27246	27277	27307	
0.75	27337	27367	27397	27427	27457	27487	27517	27547	27577	27607	30
0.76	27637	27667	27697	27726	27756	27786	27816	27845	27875	27905	
0.77	27935	27964	27994	28023	28053	28082	28112	28142	28171	28201	
0.78	28230	28260	28289	28318	28347	28377	28406	28435	28465	28494	
0.79	28524	28553	28582	28611	28640	28669	28698	28727	28756	28785	
0.80	28814	28843	28872	28901	28930	28958	28987	29016	29045	29074	29

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3
36	3.6	7.2	10.8	14.4	18.0	21.6	25.2	28.8	32.4
35	3.5	7.0	10.5	14.0	17.5	21.0	24.5	28.0	31.5
34	3.4	6.8	10.2	13.6	17.0	20.4	23.8	27.2	30.6
33	3.3	6.6	9.9	13.2	16.5	19.8	23.1	26.4	29.7
32	3.2	6.4	9.6	12.8	16.0	19.2	22.4	25.6	28.8
31	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9
30	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1

TABLE IV.—*Continued*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
0.81	29103	29132	29160	29189	29217	29246	29274	29303	29332	29360	
0.82	29389	29417	29446	29474	29502	29531	29559	29588	29616	29645	
0.83	29673	29701	29729	29757	29785	29814	29842	29870	29898	29926	
0.84	29954	29982	30010	30038	30066	30094	30122	30150	30178	30206	28
0.85	30234	30261	30289	30317	30344	30372	30400	30427	30455	30483	
0.86	30510	30538	30565	30593	30620	30648	30675	30702	30730	30757	
0.87	30785	30812	30839	30866	30894	30921	30948	30975	31002	31030	
0.88	31057	31084	31111	31138	31165	31192	31219	31246	31273	31300	27
0.89	31327	31353	31380	31407	31433	31460	31487	31514	31540	31567	
0.90	31594	31620	31647	31673	31700	31726	31753	31780	31806	31832	
0.91	31859	31885	31911	31937	31964	31990	32016	32042	32069	32095	
0.92	32121	32147	32173	32199	32225	32251	32277	32303	32329	32355	26
0.93	32381	32407	32433	32459	32484	32510	32536	32562	32587	32613	
0.94	32639	32665	32690	32715	32741	32766	32792	32818	32843	32869	
0.95	32894	32919	32945	32970	32995	33021	33046	33071	33096	33122	
0.96	33147	33172	33197	33222	33247	33272	33297	33322	33347	33373	25
0.97	33398	33422	33447	33472	33497	33521	33546	33571	33596	33621	
0.98	33646	33670	33695	33719	33744	33768	33793	33817	33842	33867	
0.99	33891	33915	33940	33964	33988	34013	34037	34061	34086	34110	
1.00	34134	34158	34182	34206	34230	34255	34279	34303	34327	34351	24
1.01	34375	34399	34423	34446	34470	34494	34518	34542	34566	34590	
1.02	34613	34637	34661	34684	34708	34731	34755	34778	34802	34826	
1.03	34849	34873	34896	34919	34943	34966	34989	35013	35036	35059	
1.04	35083	35106	35129	35152	35175	35198	35221	35245	35268	35291	23
1.05	35314	35337	35360	35382	35405	35428	35451	35474	35497	35520	
1.06	35543	35565	35588	35610	35633	35656	35678	35701	35724	35746	
1.07	35769	35791	35814	35836	35858	35881	35903	35926	35948	35970	
1.08	35993	015	037	059	081	103	125	148	170	192	22
1.09	36214	236	258	280	302	324	345	367	389	411	
1.10	433	455	477	498	520	541	563	585	607	628	
1.11	650	671	693	714	735	757	778	800	821	843	
1.12	864	885	906	928	949	970	991	012	034	055	
1.13	37076	097	118	139	160	181	202	223	244	265	21
1.14	286	306	327	348	368	389	410	430	451	472	
1.15	493	513	534	554	574	595	615	636	656	677	
1.16	697	718	738	758	778	798	819	839	859	880	
1.17	900	920	940	960	980	000	020	040	060	080	20
1.18	38100	120	139	159	179	199	218	238	258	278	
1.19	298	317	337	356	376	395	415	434	454	473	
1.20	493	512	531	551	570	589	609	628	647	667	

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1
28	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
27	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
26	2.6	5.2	7.8	10.4	13.0	15.6	18.2	20.8	23.4
25	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5
24	2.4	4.8	7.2	9.6	12.0	14.4	16.8	19.2	21.6
23	2.3	4.6	6.9	9.2	11.5	13.8	16.1	18.4	20.7
22	2.2	4.4	6.6	8.8	11.0	13.2	15.4	17.6	19.8
21	2.1	4.2	6.3	8.4	10.5	12.6	14.7	16.8	18.9
20	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1

TABLE IV.—*Continued*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
1.21	38686	705	724	743	762	781	800	819	838	857	19
1.22	876	895	914	933	952	971	990				
								008	027	046	
1.23	39065	084	102	121	139	158	177	195	214	232	
1.24	251	270	288	306	324	343	361	380	398	417	
1.25	435	453	471	489	507	525	544	562	580	598	
1.26	617	634	652	670	688	706	724	742	760	778	18
1.27	796	813	831	849	866	884	902	920	937	955	
1.28	973	990									
			008	025	042	060	077	095	112	130	
1.29	40147	165	182	199	216	233	251	268	285	303	
1.30	320	337	354	371	388	405	422	439	456	473	17
1.31	490	507	524	540	557	574	591	608	625	641	
1.32	658	675	692	709	725	742	758	775	792	808	
1.33	825	841	857	873	889	906	922	938	955	971	
1.34	987										
		004	020	036	052	068	084	101	117	133	
1.35	41149	165	181	197	213	229	245	261	277	292	16
1.36	308	324	340	355	371	387	403	418	434	450	
1.37	466	481	497	512	527	543	558	574	590	605	
1.38	621	637	652	667	683	698	713	728	744	759	
1.39	774	789	804	819	834	849	864	879	894	909	15
1.40	924	939	954	969	984	998					
							013	028	043	058	
1.41	42073	088	102	117	131	146	161	175	190	205	
1.42	220	234	248	263	277	292	306	321	335	350	
1.43	364	378	393	407	421	435	449	464	478	492	
1.44	507	521	535	549	563	577	591	605	619	633	14
1.45	647	661	675	688	702	716	730	744	758	772	
1.46	785	799	813	826	840	854	867	881	895	908	
1.47	922	935	949	962	975	989					
							002	016	029	043	
1.48	43056	069	083	096	109	122	136	149	162	175	
1.49	189	202	215	228	241	254	267	280	293	306	13
1.50	319	332	345	358	371	383	396	409	422	435	
1.51	448	460	473	486	498	511	524	536	549	562	
1.52	574	587	599	612	624	637	649	662	674	687	
1.53	699	711	724	736	748	760	773	785	797	810	
1.54	822	834	846	858	870	882	894	906	919	931	12
1.55	943	955	967	978	990						
						002	014	026	038	050	
1.56	44062	074	085	097	109	120	132	144	156	167	
1.57	179	191	202	214	225	237	248	260	271	283	
1.58	295	306	317	329	340	351	363	374	385	397	
1.59	408	419	430	442	453	464	475	486	498	509	

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1
18	1.8	3.6	5.4	7.2	9.0	10.8	12.6	14.4	16.2
17	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4
15	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5
14	1.4	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6
13	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9

TABLE IV.—*Continued*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
1.60	44520	531	542	553	564	575	586	597	608	619	11
1.61	630	641	652	662	673	684	695	706	717	727	
1.62	738	749	760	770	781	791	802	813	823	834	
1.63	845	855	866	876	887	897	908	918	929	939	
1.64	950	960	970	980	991	001	011	022	032	042	
1.65	45053	063	073	083	093	103	114	124	134	144	
1.66	154	164	174	184	194	204	214	224	234	244	10
1.67	254	264	274	283	293	303	313	323	332	342	
1.68	352	362	371	381	391	400	410	419	429	439	
1.69	449	458	467	477	486	496	505	515	524	534	
1.70	543	553	562	571	581	590	599	609	618	627	
1.71	637	646	655	664	673	682	692	701	710	719	
1.72	728	737	746	755	764	773	782	791	800	809	9
1.73	818	827	836	845	854	863	871	880	889	898	
1.74	907	916	924	933	942	950	959	968	977	985	
1.75	994	003	011	020	028	037	045	054	062	071	
1.76	46080	088	096	105	113	121	130	138	147	155	
1.77	164	172	180	188	196	205	213	221	230	238	
1.78	246	254	262	270	279	287	295	303	311	319	
1.79	327	335	343	351	359	367	375	383	391	399	8
1.80	407	415	423	430	438	446	454	462	469	477	
1.81	485	493	500	508	516	523	531	539	547	554	
1.82	562	570	577	585	592	600	607	615	622	630	
1.83	638	645	652	660	667	674	682	689	697	704	
1.84	712	719	726	733	741	748	755	762	770	777	
1.85	784	791	798	806	813	820	827	834	841	849	
1.86	856	863	870	877	884	891	898	905	912	919	7
1.87	926	933	939	946	953	960	967	974	981	988	
1.88	995	001	008	015	021	028	035	042	049	055	
1.89	47062	069	075	082	088	095	102	108	115	122	
1.90	128	135	141	148	154	161	167	174	180	187	
1.91	193	200	206	212	219	225	231	238	244	251	
1.92	257	263	270	276	282	288	294	301	307	313	
1.93	320	326	332	338	344	350	356	362	369	375	
1.94	381	387	393	399	405	411	417	423	429	435	6
1.95	441	447	453	459	465	471	476	482	488	494	
1.96	500	506	512	517	523	529	535	541	546	552	
1.97	558	564	569	575	581	586	592	598	603	609	
1.98	615	620	626	631	637	643	648	654	659	665	
1.99	670	676	681	687	692	698	703	709	714	719	
2.00	725	730	735	741	746	752	757	762	768	772	
2.01	778	784	789	794	799	804	810	815	820	826	
2.02	831	836	841	846	851	856	862	867	872	877	
2.03	882	887	892	897	902	907	912	917	922	927	5
2.04	932	937	942	947	952	957	962	967	972	977	

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
10	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
8	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
6	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4

TABLE IV.—*Continued*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
2.05	47982	987	991	996	001	006	011	015	020	025	
2.06	48030	035	039	044	049	054	058	063	068	073	
2.07	077	082	087	091	096	100	105	110	114	119	
2.08	124	128	133	137	142	146	151	155	160	165	
2.09	169	173	178	182	187	191	196	200	205	209	
2.10	214	218	222	227	231	235	240	244	248	253	
2.11	257	261	266	270	274	278	283	287	291	295	
2.12	300	304	308	312	316	320	325	329	333	337	
2.13	341	345	350	354	358	362	366	370	374	378	
2.14	382	386	390	394	398	402	406	410	414	418	4
2.15	422	426	430	434	438	442	446	450	453	457	
2.16	461	465	469	473	477	480	484	488	492	496	
2.17	500	503	507	511	515	518	522	526	530	533	
2.18	537	541	544	548	552	555	559	563	566	570	
2.19	574	577	581	584	588	592	595	599	602	606	
2.20	610	613	617	620	624	627	631	634	638	641	
2.21	645	648	652	655	658	662	665	669	672	676	
2.22	679	682	686	689	692	696	699	702	706	709	
2.23	713	716	719	722	726	729	732	736	739	742	
2.24	745	749	752	755	758	761	765	768	771	774	
2.25	778	781	784	787	790	793	796	799	803	806	
2.26	809	812	815	818	821	824	827	830	833	837	
2.27	840	843	846	849	852	855	858	861	864	867	3
2.28	870	872	875	878	881	884	887	890	893	896	
2.29	899	902	905	907	910	913	916	919	922	925	
2.30	928	930	933	936	939	942	944	947	950	953	
2.31	956	958	961	964	966	969	972	975	977	980	
2.32	983	986	988	991	994	996	999	002	004	007	
2.33	49010	012	015	017	020	023	025	028	031	033	
2.34	036	038	041	043	046	048	051	054	056	059	
2.35	061	064	066	069	071	074	076	079	081	084	
2.36	086	089	092	094	096	098	101	103	106	108	
2.37	111	113	115	118	120	122	125	127	130	132	
2.38	134	137	139	141	144	146	148	151	153	155	
2.39	158	160	162	164	167	169	171	173	176	178	
2.40	180	182	185	187	189	191	193	196	198	200	
2.41	202	205	207	209	211	213	215	217	220	222	
2.42	224	226	228	230	232	234	237	239	241	243	
2.43	245	247	249	251	253	255	257	259	261	264	
2.44	266	268	270	272	274	276	278	280	282	284	2
2.45	286	288	290	292	294	295	297	299	301	303	
2.46	305	307	309	311	313	315	317	319	321	323	
2.47	324	326	328	330	332	334	336	337	339	341	
2.48	343	345	347	349	350	352	354	356	358	359	
2.49	361	363	365	367	368	370	372	374	375	377	
2.5	379	396	413	430	446	461	477	492	506	520	16
2.6	534	547	560	573	585	598	609	621	632	643	12
2.7	653	664	674	683	693	702	711	720	728	736	9
2.8	744	752	760	767	774	781	788	795	801	807	7

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3

TABLE IV.—*Continued.*

x/σ	0	1	2	3	4	5	6	7	8	9	Δ
2.9	49813	819	825	831	836	841	846	851	856	861	5
3.0	865	869	873	878	882	886	889	893	897	900	4
3.1	903	906	910	913	916	918	921	924	926	929	3
3.2	931	934	936	938	940	942	944	946	948	950	2
3.3	952	953	955	957	958	960	961	962	964	965	1
3.4	966	968	969	970	971	972	973	974	975	976	1
3.5	977	978	978	979	980	981	981	982	983	983	1
3.6	984	985	985	986	986	987	987	988	988	989	1
3.7	989	990	990	990	991	991	992	992	992	992	0
3.8	993	993	993	994	994	994	994	995	995	995	0
3.9	995	995	996	996	996	996	996	996	997	997	0
4	997	998	999	999	999	000	000	000	000	000	0

PROPORTIONAL PARTS, Δ

	1	2	3	4	5	6	7	8	9
5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
4	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6
3	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7
2	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

TABLE V.—REDUCTION FROM COMMON TO METRIC SYSTEM

(a) Inches to Millimeters

	1	2	3	4	5	6	7	8	9
...	25.40	50.80	76.20	101.60	127.00	152.40	177.80	203.20	228.60
10	279.40	304.80	330.19	355.59	380.99	406.39	431.79	457.19	482.59
20	533.39	558.79	584.19	609.59	634.99	660.39	685.79	711.19	736.59
30	787.39	812.79	838.19	863.59	888.99	914.39	939.78	965.18	990.58
40	1041.4	1066.8	1092.2	1117.6	1143.0	1168.4	1193.8	1219.2	1244.6
50	1295.4	1320.8	1346.2	1371.6	1397.0	1422.4	1447.8	1473.2	1498.6
60	1549.4	1574.8	1600.2	1625.6	1651.0	1676.4	1701.8	1727.2	1752.6
70	1803.4	1828.8	1854.2	1879.6	1905.0	1930.4	1955.8	1981.2	2006.6
80	2057.4	2082.8	2108.2	2133.6	2159.0	2184.4	2209.8	2235.2	2260.6
90	2311.4	2336.8	2362.2	2387.6	2413.0	2438.4	2463.8	2489.2	2514.6

Twelfths				Sixteenths							
1/12	2.12	7/12	14.82	1/16	1.59	5/16	7.94	9/16	14.29	13/16	20.64
2/12	4.23	8/12	16.93	1/8	3.17	3/8	9.52	5/8	15.87	7/8	22.22
3/12	6.35	9/12	19.05	3/16	4.76	7/16	11.11	11/16	17.46	15/16	23.81
4/12	8.47	10/12	21.17	1/4	6.35	1/2	12.70	3/4	19.05	1	25.40
5/12	10.58	11/12	23.28								
6/12	12.70	12/12	25.40								

(b) Avoirdupois Ounces into Grams

1 ounce = 28.349527 grams

Oz.	0	1	2	3	4	5	6	7	8	9
0	0	28.4	56.7	85.0	113.4	141.7	170.1	198.4	226.8	255.1
10	283.5	311.8	340.2	368.5	396.9	425.2	453.6	481.9	510.3	538.6
20	567.0	595.3	623.7	652.0	680.4	708.7	737.1	765.4	793.8	822.1
30	850.5	878.8	907.2	935.5	963.9	992.2	1020.6	1048.9	1077.3	1105.6
40	1134.0	1162.3	1190.7	1219.0	1247.4	1275.7	1304.1	1332.4	1360.8	1389.1
50	1417.5	1445.8	1474.2	1502.5	1530.9	1559.2	1587.6	1615.9	1644.3	1672.6
60	1701.0	1729.3	1757.7	1786.0	1814.4	1842.7	1871.1	1899.4	1927.8	1956.1
70	1984.5	2012.8	2041.2	2069.5	2097.9	2126.2	2154.6	2182.9	2211.3	2239.6
80	2268.0	2296.3	2324.7	2353.0	2381.4	2409.7	2438.1	2466.4	2494.8	2523.1
90	2551.5	2579.8	2608.2	2636.5	2664.9	2693.2	2721.6	2749.9	2778.3	2806.6

(c) Avoirdupois Pounds into Grams

1 pound = 0.4535924 kilogram

Lb.	0	1	2	3	4	5	6	7	8	9
0	0	454	907	1361	1814	2268	2722	3175	3629	4082
10	4536	4990	5443	5897	6350	6804	7258	7711	8165	8618
20	9072	9526	9979	10433	10886	11340	11794	12247	12701	13154
30	13608	14062	14515	14969	15422	15876	16329	16783	17237	17690
40	18144	18597	19051	19505	19958	20412	20865	21319	21773	22226
50	22680	23133	23587	24041	24494	24948	25401	25855	26309	26762
60	27216	27669	28123	28577	29030	29484	29937	30391	30845	31298
70	31752	32205	32659	33113	33566	34020	34473	34927	35381	35834
80	36288	36741	37195	37649	38102	38556	39009	39463	39917	40370
90	40824	41277	41731	42185	42638	43092	43545	43999	44453	44906

TABLE VI.—MINUTES AND SECONDS IN DECIMALS OF A DEGREE

′	°	′	°	′	°	″	°	″	°	″	°
1	.016666	21	.350000	41	.683333	1	.000278*	21	.005833	41	.011389
2	.033333	22	.366666	42	.700000	2	.000556	22	.006111	42	.011667
3	.050000	23	.383333	43	.716666	3	.000833	23	.006389	43	.011944
4	.066666	24	.400000	44	.733333	4	.001111	24	.006667	44	.012222
5	.083333	25	.416666	45	.750000	5	.001389	25	.006944	45	.012500
6	.100000	26	.433333	46	.766666	6	.001667	26	.007222	46	.012778
7	.116666	27	.450000	47	.783333	7	.001944	27	.007500	47	.013056
8	.133333	28	.466666	48	.800000	8	.002222	28	.007778	48	.013333
9	.150000	29	.483333	49	.816666	9	.002500	29	.008056	49	.013611
10	.166666	30	.500000	50	.833333	10	.002778	30	.008333	50	.013889
11	.183333	31	.516666	51	.850000	11	.003056	31	.008611	51	.014167
12	.200000	32	.533333	52	.866666	12	.003333	32	.008889	52	.014444
13	.216666	33	.550000	53	.883333	13	.003611	33	.009167	53	.014722
14	.233333	34	.566666	54	.900000	14	.003889	34	.009444	54	.015000
15	.250000	35	.583333	55	.916666	15	.004167	35	.009722	55	.015278
16	.266666	36	.600000	56	.933333	16	.004444	36	.010000	56	.015556
17	.283333	37	.616666	57	.950000	17	.004722	37	.010278	57	.015833
18	.300000	38	.633333	58	.966666	18	.005000	38	.010556	58	.016111
19	.316666	39	.650000	59	.983333	19	.005278	39	.010833	59	.016389
20	.333333	40	.666666	60	1.000000	20	.005556	40	.011111	60	.016667

* .0002777778

TABLE VII.—FIRST TO SIXTH POWERS OF INTEGERS FROM 1 TO 30

Powers					
First	Second	Third	Fourth	Fifth	Sixth
1	1	1	1	1	1
2	4	8	16	32	64
3	9	27	81	243	729
4	16	64	256	1024	4096
5	25	125	625	3125	15625
6	36	216	1296	7776	46656
7	49	343	2401	16807	117649
8	64	512	4096	32768	262144
9	81	729	6561	59049	531441
10	100	1000	10000	100000	1000000
11	121	1331	14641	161051	1771561
12	144	1728	20736	248832	2985984
13	169	2197	28561	371293	4826809
14	196	2744	38416	537824	7529536
15	225	3375	50625	759375	11390625
16	256	4096	65536	1048576	16777216
17	289	4913	83521	1419857	24137569
18	324	5832	104976	1889568	34012224
19	361	6859	130321	2476099	47045881
20	400	8000	160000	3200000	64000000
21	441	9261	194481	4084101	85766121
22	484	10648	234256	5153632	113379904
23	529	12167	279841	6436343	148035889
24	576	13824	331776	7962624	191102976
25	625	15625	390625	9765625	244140625
26	676	17576	456976	11881376	308915776
27	729	19683	531441	14348907	387420489
28	784	21952	614656	17210368	481890304
29	841	24389	707281	20511149	594823321
30	900	27000	810000	24300000	729000000

TABLE VIII.—FACTORIALS OF INTEGERS 1 TO 20

1	1	11	39,916,800
2	2	12	479,001,600
3	6	13	6,227,020,800
4	24	14	87,178,291,200
5	120	15	1,307,674,368,000
6	720	16	20,922,789,888,000
7	5,040	17	355,687,428,096,000
8	40,320	18	6,402,373,705,728,000
9	362,880	19	121,645,100,408,832,000
10	3,628,800	20	2,432,902,008,176,640,000

TABLE IX.—FACTORS FOR USE IN COMPUTING THE PROBABLE ERRORS OF THE MEANS AND OF THE STANDARD DEVIATIONS

$$PE_M = \frac{0.6745}{\sqrt{N}}\sigma. \quad PE_\sigma = \frac{0.6745}{\sqrt{2N}}\sigma$$

To obtain the probable errors the factors in this table have to be multiplied by σ.

N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$	N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$	N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$
5	.301641	.213292	56	.090132	.063733	106	.065512	.046324
6	.275359	.194708	57	.089338	.063172	107	.065205	.046107
7	.254933	.180265	58	.088565	.062625	108	.064903	.045893
8	.238468	.168622	59	.087811	.062092	109	.064604	.045682
9	.224830	.158979	60	.087076	.061572	110	.064310	.045474
10	.213292	.150820	61	.086360	.061065	111	.064020	.045269
11	.203366	.143802	62	.085660	.060571	112	.063733	.045066
12	.194708	.137680	63	.084978	.060088	113	.063451	.044866
13	.187070	.132278	64	.084311	.059617	114	.063172	.044669
14	.180265	.127467	65	.083660	.059157	115	.062896	.044475
15	.174153	.123144	66	.083024	.058707	116	.062625	.044282
16	.168622	.119234	67	.082402	.058267	117	.062357	.044093
17	.163588	.115674	68	.081794	.057837	118	.062092	.043906
18	.158979	.112415	69	.081199	.057416	119	.061830	.043721
19	.154739	.109417	70	.080617	.057005	120	.061572	.043538
20	.150820	.106646	71	.080047	.056602	121	.061317	.043358
21	.147186	.104076	72	.079489	.056207	122	.061065	.043180
22	.143802	.101683	73	.078943	.055821	123	.060817	.043004
23	.140641	.099448	74	.078408	.055443	124	.060571	.042830
24	.137680	.097354	75	.077883	.055072	125	.060328	.042659
25	.134898	.095387	76	.077369	.054708	126	.060088	.042489
26	.132278	.093535	77	.076865	.054352	127	.059851	.042321
27	.129806	.091786	78	.076371	.054002	128	.059617	.042156
28	.127467	.090132	79	.075886	.053660	129	.059386	.041992
29	.125250	.088565	80	.075410	.053323	130	.059157	.041830
30	.123144	.087076	81	.074943	.052993	131	.058930	.041670
31	.121142	.085660	82	.074485	.052669	132	.058707	.041512
32	.119234	.084311	83	.074035	.052351	133	.058486	.041356
33	.117414	.083024	84	.073593	.052038	134	.058267	.041201
34	.115674	.081794	85	.073159	.051731	135	.058051	.041048
35	.114010	.080617	86	.072732	.051429	136	.057837	.040897
36	.112415	.079489	87	.072313	.051133	137	.057626	.040747
37	.110885	.078408	88	.071901	.050842	138	.057416	.040599
38	.109417	.077369	89	.071496	.050555	139	.057209	.040453
39	.108005	.076371	90	.071097	.050273	140	.057005	.040308
40	.106646	.075410	91	.070706	.049996	141	.056802	.040165
41	.105338	.074485	92	.070320	.049724	142	.056602	.040024
42	.104076	.073593	93	.069941	.049456	143	.056404	.039883
43	.102859	.072732	94	.069568	.049192	144	.056207	.039745
44	.101683	.071901	95	.069201	.048933	145	.056013	.039607
45	.100547	.071097	96	.068840	.048677	146	.055821	.039472
46	.099448	.070320	97	.068484	.048426	147	.055631	.039337
47	.098384	.069568	98	.068134	.048178	148	.055443	.039204
48	.097354	.068840	99	.067789	.047934	149	.055256	.039072
49	.096356	.068134	100	.067449	.047694	150	.055072	.038942
50	.095387	.067449	101	.067114	.047457	151	.054889	.038813
51	.094447	.066784	102	.066784	.047224	152	.054708	.038685
52	.093535	.066139	103	.066459	.046994	153	.054529	.038558
53	.092648	.065512	104	.066139	.046767	154	.054352	.038433
54	.091786	.064903	105	.065823	.046544	155	.054176	.038308
55	.090948	.064310						

TABLE IX.—*Continued*

N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$	N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$	N	$\frac{0.6745}{\sqrt{N}}$	$\frac{0.6745}{\sqrt{2N}}$
156	.054002	.038185	188	.049192	.034784	220	.045474	.032155
157	.053830	.038064	189	.049062	.034692	221	.045371	.032082
158	.053660	.037943	190	.048933	.034601	222	.045269	.032010
159	.053491	.037823	191	.048804	.034510	223	.045167	.031938
160	.053323	.037705	192	.048677	.034420	224	.045066	.031867
161	.053157	.037588	193	.048551	.034331	225	.044966	.031796
162	.052993	.037472	194	.048426	.034242	226	.044866	.031725
163	.052830	.037357	195	.048301	.034154	227	.044767	.031655
164	.052669	.037242	196	.048178	.034067	228	.044669	.031586
165	.052509	.037129	197	048055	.033980	229	.044572	.031517
166	.052351	.037017	198	.047934	.033894	230	.044475	.031448
167	.052194	.036906	199	.047813	.033809	231	.044378	.031380
168	.052038	.036796	200	.047694	.033724	232	.044282	.031312
169	.051884	.036687	201	.047575	.033641	233	.044187	.031245
170	.051731	.036579	202	.047457	.033557	234	.044093	.031178
171	.051580	.036472	203	.047340	.033474	235	.043999	.031112
172	.051429	036366	204	.047224	.033392	236	.043906	.031046
173	.051281	.036261	205	.047108	.033311	237	.043813	.030980
174	.051133	.036156	206	.046994	.033230	238	.043721	.030915
175	.050987	.036053	207	.046880	.033149	239	.043629	.030850
176	.050842	.035950	208	.046767	.033070	240	.043538	.030786
177	.050698	.035849	209	.046655	.032990	241	.043448	.030722
178	.050555	.035748	210	.046544	.032912	242	.043358	.030659
179	.050414	.035648	211	.046434	.032834	243	.043269	.030595
180	.050273	.035549	212	.046324	.032756	244	.043180	.030533
181	.050134	.035450	213	.046215	.032679	245	.043092	.030470
182	.049996	.035353	214	.046107	.032603	246	.043004	.030408
183	.049860	.035256	215	.046000	.032527	247	.042917	.030347
184	.049724	.035160	216	.045893	.032451	248	.042830	.030285
185	.049589	.035065	217	.045787	.032377	249	.042744	.030225
186	.049456	.034971	218	.045682	.032302	250	.042658	.030164
187	.049324	.034877	219	.045578	.032228			

TABLE X.—ODDS AGAINST THE OCCURRENCE OF A DEVIATION IN TERMS OF THE STANDARD DEVIATION (OF AN INDIVIDUAL IN A DISTRIBUTION OR OF A SAMPLE MEAN) AS GREAT OR GREATER IN A LARGE COLLECTION OF SAMPLES TAKEN FROM A HOMOGENEOUS POPULATION. NEWLY COMPUTED

x/σ	Chances	x/σ	Chances	x/σ	Chances
1.0	2.15 to 1	2.1	26.99 to 1	3.1	515.74 to 1
1.1	2.69	2.2	34.96	3.2	726.70
1.2	3.35	2.3	45.62	3.3	1033.34
1.3	4.17	2.4	59.99	3.4	1483.12
1.4	5.19	2.5	79.52	3.5	2148.61
1.5	6.48	2.6	106.27	3.6	3141.68
1.6	8.12	2.7	143.22	3.7	4637.22
1.7	10.22	2.8	194.69	3.8	6914.63
1.8	12.92	2.9	266.98	3.9	10394.01
1.9	16.41	3.0	369.40	4.0	15771.87
2.0	20.98				

TABLE XI.—GIVING FOR DEVIATIONS IN MULTIPLES OF THE PROBABLE ERROR FROM 1.0 TO 5.0 IN COLUMN 1, THE PROBABLE OCCURRENCE OF A DEVIATION AS GREAT OR GREATER IN 100 TRIALS (COLUMN 2); ALSO ODDS AGAINST THE OCCURRENCE OF A DEVIATION AS GREAT OR GREATER IN 100 TRIALS, x TO 1 (COLUMN 3)

1	2	3	1	2	3
1.0	50.0	1 to 1	3.0	4.30	22.2 to 1
1.2	41.8	1.39	3.1	3.65	26.4
1.4	34.5	1.90	3.2	3.09	31.4
1.6	28.1	2.57	3.3	2.60	37.4
1.8	22.5	3.45	3.4	2.18	44.8
2.0	17.73	4.64	3.5	1.82	53.8
2.1	15.67	5.38	3.6	1.52	64.9
2.2	13.78	6.25	3.7	1.26	78.5
2.3	12.08	7.28	3.8	1.04	95.4
2.4	10.55	8.48	3.9	.853	116.3
2.5	9.18	9.90	4.0	.698	142.3
2.6	7.95	11.58	4.2	.461	215.8
2.7	6.86	13.58	4.4	.300	332.4
2.8	5.89	15.96	4.6	.192	520.4
2.9	5.05	18.82	4.8	.121	828.3
			5.0	.0745	1341.0

TABLE XII.—SHOWING THE PROBABILITY (P) THAT, WITH A GIVEN "DEGREE OF FREEDOM," (N') THE χ^2 VALUE OBTAINED IN THE COMPARISON OF THE DISTRIBUTION OF A SAMPLE WITH THAT OF A THEORETICAL SERIES INDICATES THAT THE SAMPLE BELONGS TO OR HAS ARISEN OUT OF SUCH SERIES

N'	$P = 0.99$	0.98	0.95	0.90	0.80	0.70	0.50	0.30	0.20	0.10	0.05	0.02	0.01
1	0.000157	0.000628	0.00393	0.0158	0.0642	0.148	0.455	1.074	1.642	2.706	3.841	5.412	6.635
2	0.0201	0.0404	0.103	0.211	0.446	0.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.341
4	0.297	0.429	0.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277
5	0.554	0.752	1.145	1.610	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086
6	0.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666
10	2.558	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.812	18.549	21.026	24.054	26.217
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.201	30.144	33.687	36.191
20	8.260	9.237	10.851	12.443	14.578	16.266	19.337	22.775	25.038	28.412	31.410	35.020	37.566
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	40.113	44.140	46.963
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588
30	14.953	16.306	18.493	20.599	23.364	25.508	29.336	33.530	36.250	40.256	43.773	47.962	50.892

For larger values of N', the expression $\sqrt{2\chi^2} - \sqrt{2N' - 1}$ may be used as a normal deviate with unit standard error.

TABLE XIII.—TABLES OF VALUES OF *F*, BEING THE RATIO OF THE LARGER MEAN VARIANCE BETWEEN SAMPLES TO THE SMALLER MEAN VARIANCE WITHIN SAMPLES, INDICATING, FOR THE VARIOUS "DEGREES OF FREEDOM" OF THE TWO SERIES, OF WHAT SIZE THE RATIO MUST BE TO BE SIGNIFICANT OF A REAL LACK OF HOMOGENEITY BETWEEN THE SAMPLES. ALSO VALUES OF *t*.

Degrees of Freedom for Smaller Mean Square	Degrees of Freedom for Greater Mean Square														Values at *t*
	1	2	3	4	5	6	7	8	10	12	16	24	50	∞	
1	161.45 **4052.10**	199.50 **4999.03**	215.72 **5403.49**	224.57 **5625.14**	230.17 **5764.08**	233.97 **5859.39**	236.75 **5928.00**	238.89 **5981.34**	242.00 **6056.00**	243.91 **6105.83**	246.50 **6169.00**	249.04 **6234.16**	251.80 **6302.00**	254.32 **6366.48**	12.706 **63.657**
2	18.51 **98.49**	19.00 **99.01**	19.16 **99.17**	19.25 **99.25**	19.30 **99.30**	19.33 **99.33**	19.36 **99.34**	19.37 **99.36**	19.39 **99.40**	19.41 **99.42**	19.43 **99.44**	19.45 **99.46**	19.47 **99.48**	19.50 **99.50**	4.303 **9.925**
3	10.13 **34.12**	9.55 **30.81**	9.28 **29.46**	9.12 **28.71**	9.01 **28.24**	8.94 **27.91**	8.88 **27.67**	8.84 **27.49**	8.78 **27.23**	8.74 **27.05**	8.69 **26.83**	8.64 **26.60**	8.58 **26.35**	8.53 **26.12**	3.182 **5.841**
4	7.71 **21.20**	6.94 **18.00**	6.59 **16.69**	6.39 **15.98**	6.26 **15.52**	6.16 **15.21**	6.09 **14.98**	6.04 **14.80**	5.96 **14.54**	5.91 **14.37**	5.84 **14.15**	5.77 **13.93**	5.70 **13.69**	5.63 **13.46**	2.776 **4.604**
5	6.61 **16.26**	5.79 **13.27**	5.41 **12.06**	5.19 **11.39**	5.05 **10.97**	4.95 **10.67**	4.88 **10.45**	4.82 **10.27**	4.74 **10.05**	4.68 **9.89**	4.60 **9.68**	4.53 **9.47**	4.44 **9.24**	4.36 **9.02**	2.571 **4.032**
6	5.99 **13.74**	5.14 **10.92**	4.76 **9.78**	4.53 **9.15**	4.39 **8.75**	4.28 **8.47**	4.21 **8.26**	4.15 **8.10**	4.06 **7.87**	4.00 **7.72**	3.92 **7.52**	3.84 **7.31**	3.75 **7.09**	3.67 **6.88**	2.447 **3.707**
7	5.59 **12.25**	4.74 **9.55**	4.35 **8.45**	4.12 **7.85**	3.97 **7.46**	3.87 **7.19**	3.79 **7.00**	3.73 **6.84**	3.63 **6.62**	3.57 **6.47**	3.49 **6.27**	3.41 **6.07**	3.32 **5.85**	3.23 **5.65**	2.365 **3.499**
8	5.32 **11.26**	4.46 **8.65**	4.07 **7.59**	3.84 **7.01**	3.69 **6.63**	3.58 **6.37**	3.50 **6.19**	3.44 **6.03**	3.34 **5.82**	3.28 **5.67**	3.20 **5.48**	3.12 **5.28**	3.03 **5.06**	2.93 **4.86**	2.306 **3.355**
9	5.12 **10.56**	4.26 **8.02**	3.86 **6.99**	3.63 **6.42**	3.48 **6.06**	3.37 **5.80**	3.29 **5.62**	3.23 **5.47**	3.13 **5.26**	3.07 **5.11**	2.98 **4.92**	2.90 **4.73**	2.80 **4.51**	2.71 **4.31**	2.262 **3.250**
10	4.96 **10.04**	4.10 **7.56**	3.71 **6.55**	3.48 **5.99**	3.33 **5.64**	3.22 **5.39**	3.14 **5.21**	3.07 **5.06**	2.97 **4.85**	2.91 **4.71**	2.82 **4.52**	2.74 **4.33**	2.64 **4.12**	2.54 **3.91**	2.228 **3.169**

Degrees of Freedom for Smaller Mean Square

11	4.84 **9.65**	3.98 **7.20**	3.59 **6.22**	3.36 **5.67**	3.20 **5.32**	3.09 **5.07**	3.01 **4.88**	2.95 **4.74**	2.86 **4.54**	2.79 **4.40**	2.70 **4.21**	2.61 **4.02**	2.50 **3.80**	2.40 **3.60**	2.201 **3.106**
12	4.75 **9.33**	3.88 **6.93**	3.49 **5.95**	3.26 **5.41**	3.11 **5.06**	3.00 **4.82**	2.92 **4.65**	2.85 **4.50**	2.76 **4.30**	2.69 **4.16**	2.60 **3.98**	2.50 **3.78**	2.40 **3.56**	2.30 **3.36**	2.179 **3.055**
13	4.67 **9.07**	3.80 **6.70**	3.41 **5.74**	3.18 **5.20**	3.02 **4.86**	2.92 **4.62**	2.84 **4.44**	2.77 **4.30**	2.67 **4.10**	2.60 **3.96**	2.51 **3.78**	2.42 **3.59**	2.32 **3.37**	2.21 **3.16**	2.160 **3.012**
14	4.60 **8.86**	3.74 **6.51**	3.34 **5.56**	3.11 **5.03**	2.96 **4.69**	2.85 **4.46**	2.77 **4.28**	2.70 **4.14**	2.60 **3.94**	2.53 **3.80**	2.44 **3.62**	2.35 **3.43**	2.24 **3.21**	2.13 **3.00**	2.145 **2.977**
15	4.54 **8.68**	3.68 **6.36**	3.29 **5.42**	3.06 **4.89**	2.90 **4.56**	2.79 **4.32**	2.70 **4.14**	2.64 **4.00**	2.55 **3.80**	2.48 **3.67**	2.39 **3.48**	2.29 **3.29**	2.18 **3.07**	2.07 **2.87**	2.131 **2.947**
16	4.49 **8.53**	3.63 **6.23**	3.24 **5.29**	3.01 **4.77**	2.85 **4.44**	2.74 **4.20**	2.66 **4.03**	2.59 **3.89**	2.49 **3.69**	2.42 **3.55**	2.33 **3.37**	2.24 **3.18**	2.13 **2.96**	2.01 **2.75**	2.120 **2.921**
17	4.45 **8.40**	3.59 **6.11**	3.20 **5.18**	2.96 **4.67**	2.81 **4.34**	2.70 **4.10**	2.62 **3.93**	2.55 **3.79**	2.45 **3.59**	2.38 **3.45**	2.29 **3.27**	2.19 **3.08**	2.08 **2.86**	1.96 **2.65**	2.110 **2.898**
18	4.41 **8.28**	3.55 **6.01**	3.16 **5.09**	2.93 **4.58**	2.77 **4.25**	2.66 **4.01**	2.58 **3.85**	2.51 **3.71**	2.41 **3.51**	2.34 **3.37**	2.25 **3.20**	2.15 **3.01**	2.04 **2.79**	1.92 **2.57**	2.101 **2.878**
19	4.38 **8.18**	3.52 **5.93**	3.13 **5.01**	2.90 **4.50**	2.74 **4.17**	2.63 **3.94**	2.55 **3.77**	2.48 **3.63**	2.38 **3.43**	2.31 **3.30**	2.21 **3.12**	2.11 **2.92**	2.00 **2.70**	1.88 **2.49**	2.093 **2.861**
20	4.35 **8.10**	3.49 **5.85**	3.10 **4.94**	2.87 **4.43**	2.71 **4.10**	2.60 **3.87**	2.52 **3.69**	2.45 **3.56**	2.35 **3.37**	2.28 **3.23**	2.18 **3.05**	2.08 **2.86**	1.96 **2.63**	1.84 **2.42**	2.086 **2.845**
21	4.32 **8.02**	3.47 **5.78**	3.07 **4.87**	2.84 **4.37**	2.68 **4.04**	2.57 **3.81**	2.49 **3.65**	2.42 **3.51**	2.32 **3.31**	2.25 **3.17**	2.15 **2.99**	2.05 **2.80**	1.93 **2.58**	1.81 **2.36**	2.080 **2.831**
22	4.30 **7.94**	3.44 **5.72**	3.05 **4.82**	2.82 **4.31**	2.66 **3.99**	2.55 **3.75**	2.47 **3.59**	2.40 **3.45**	2.30 **3.26**	2.23 **3.12**	2.13 **2.94**	2.03 **2.75**	1.91 **2.53**	1.78 **2.30**	2.074 **2.819**
23	4.28 **7.88**	3.42 **5.66**	3.03 **4.76**	2.80 **4.26**	2.64 **3.94**	2.53 **3.71**	2.45 **3.55**	2.38 **3.41**	2.28 **3.21**	2.20 **3.07**	2.11 **2.89**	2.00 **2.70**	1.88 **2.48**	1.76 **2.26**	2.069 **2.807**

TABLE XIII—*Continued*

Degrees of Freedom for Smaller Mean Square	Degrees of Freedom for Greater Mean Square														Values at t
	1	2	3	4	5	6	7	8	10	12	16	24	50	∞	
24	4.26 **7.82**	3.40 **5.61**	3.01 **4.72**	2.78 **4.22**	2.62 **3.90**	2.51 **3.67**	2.42 **3.50**	2.36 **3.36**	2.26 **3.17**	2.18 **3.03**	2.09 **2.85**	1.98 **2.66**	1.86 **2.44**	1.73 **2.21**	2.064 **2.797**
25	4.24 **7.77**	3.38 **5.57**	2.99 **4.68**	2.76 **4.18**	2.60 **3.86**	2.49 **3.63**	2.41 **3.46**	2.34 **3.32**	2.24 **3.13**	2.16 **2.99**	2.07 **2.81**	1.96 **2.62**	1.84 **2.40**	1.71 **2.17**	2.060 **2.787**
26	4.22 **7.72**	3.37 **5.53**	2.98 **4.64**	2.74 **4.14**	2.59 **3.82**	2.47 **3.59**	2.39 **3.42**	2.32 **3.29**	2.22 **3.09**	2.15 **2.96**	2.05 **2.78**	1.95 **2.58**	1.82 **2.36**	1.69 **2.13**	2.056 **2.779**
27	4.21 **7.68**	3.35 **5.49**	2.96 **4.60**	2.73 **4.11**	2.57 **3.78**	2.46 **3.56**	2.37 **3.39**	2.30 **3.26**	2.20 **3.06**	2.13 **2.93**	2.03 **2.74**	1.93 **2.55**	1.80 **2.33**	1.67 **2.10**	2.052 **2.771**
28	4.20 **7.64**	3.34 **5.45**	2.95 **4.57**	2.71 **4.07**	2.56 **3.75**	2.44 **3.53**	2.36 **3.36**	2.29 **3.23**	2.19 **3.03**	2.12 **2.90**	2.02 **2.71**	1.91 **2.52**	1.78 **2.30**	1.65 **2.06**	2.048 **2.763**
29	4.18 **7.60**	3.33 **5.42**	2.93 **4.54**	2.70 **4.04**	2.54 **3.73**	2.43 **3.50**	2.35 **3.33**	2.28 **3.20**	2.18 **3.00**	2.10 **2.87**	2.00 **2.68**	1.90 **2.49**	1.77 **2.27**	1.64 **2.03**	2.045 **2.756**
30	4.17 **7.56**	3.32 **5.39**	2.92 **4.51**	2.69 **4.02**	2.53 **3.70**	2.42 **3.47**	2.34 **3.30**	2.27 **3.17**	2.16 **2.98**	2.09 **2.84**	1.99 **2.66**	1.89 **2.47**	1.76 **2.24**	1.62 **2.01**	2.042 **2.750**
35	4.12 **7.42**	3.26 **5.27**	2.87 **4.40**	2.64 **3.91**	2.48 **3.59**	2.37 **3.37**	2.29 **3.19**	2.22 **3.07**	2.11 **2.87**	2.04 **2.74**	1.94 **2.56**	1.83 **2.37**	1.70 **2.13**	1.57 **1.90**	2.030 **2.724**
40	4.08 **7.31**	3.23 **5.18**	2.84 **4.31**	2.61 **3.83**	2.45 **3.51**	2.34 **3.29**	2.25 **3.12**	2.18 **2.99**	2.07 **2.80**	2.00 **2.66**	1.90 **2.48**	1.79 **2.29**	1.66 **2.05**	1.52 **1.82**	2.021 **2.704**
45	4.06 **7.23**	3.21 **5.11**	2.81 **4.25**	2.58 **3.77**	2.42 **3.45**	2.31 **3.23**	2.22 **3.07**	2.15 **2.94**	2.05 **2.74**	1.97 **2.61**	1.87 **2.43**	1.76 **2.23**	1.63 **1.99**	1.48 **1.75**	2.014 **2.690**
50	4.03 **7.17**	3.18 **5.06**	2.79 **4.20**	2.56 **3.72**	2.40 **3.41**	2.29 **3.19**	2.20 **3.02**	2.13 **2.89**	2.02 **2.70**	1.95 **2.56**	1.85 **2.38**	1.74 **2.18**	1.60 **1.94**	1.44 **1.68**	2.008 **2.678**

Degrees of Freedom for Smaller Mean Square														
60	4.00 **7.08**	3.15 **4.98**	2.76 **4.13**	2.52 **3.65**	2.37 **3.34**	2.25 **3.12**	2.17 **2.95**	2.10 **2.82**	1.99 **2.63**	1.92 **2.50**	1.81 **2.32**	1.70 **2.12**	1.56 **1.87**	1.39 **1.60**	2.000 **2.660**
70	3.98 **7.01**	3.13 **4.92**	2.74 **4.07**	2.50 **3.60**	2.35 **3.29**	2.23 **3.07**	2.14 **2.91**	2.07 **2.78**	1.97 **2.59**	1.89 **2.45**	1.79 **2.28**	1.67 **2.07**	1.53 **1.82**	1.35 **1.53**	1.994 **2.648**
80	3.96 **6.96**	3.11 **4.88**	2.72 **4.04**	2.49 **3.56**	2.33 **3.26**	2.21 **3.04**	2.12 **2.87**	2.06 **2.74**	1.95 **2.55**	1.88 **2.42**	1.77 **2.24**	1.65 **2.03**	1.51 **1.78**	1.32 **1.49**	1.990 **2.638**
90	3.95 **6.92**	3.10 **4.85**	2.71 **4.01**	2.47 **3.53**	2.32 **3.23**	2.20 **3.01**	2.11 **2.84**	2.04 **2.72**	1.93 **2.53**	1.86 **2.39**	1.76 **2.21**	1.64 **2.00**	1.49 **1.75**	1.30 **1.45**	1.987 **2.632**
100	3.94 **6.90**	3.09 **4.82**	2.70 **3.98**	2.46 **3.51**	2.30 **3.21**	2.19 **2.99**	2.10 **2.82**	2.03 **2.69**	1.92 **2.51**	1.85 **2.37**	1.75 **2.19**	1.63 **1.98**	1.48 **1.73**	1.28 **1.43**	1.984 **2.626**
125	3.92 **6.84**	3.07 **4.78**	2.68 **3.94**	2.44 **3.47**	2.29 **3.17**	2.17 **2.95**	2.08 **2.79**	2.01 **2.66**	1.90 **2.47**	1.83 **2.33**	1.72 **2.15**	1.60 **1.94**	1.45 **1.69**	1.25 **1.37**	1.979 **2.616**
150	3.90 **6.81**	3.06 **4.75**	2.66 **3.91**	2.43 **3.45**	2.27 **3.14**	2.16 **2.92**	2.07 **2.76**	2.00 **2.63**	1.89 **2.45**	1.82 **2.31**	1.71 **2.13**	1.59 **1.92**	1.44 **1.66**	1.22 **1.33**	1.976 **2.609**
200	3.89 **6.76**	3.04 **4.71**	2.65 **3.88**	2.42 **3.41**	2.26 **3.11**	2.14 **2.89**	2.05 **2.73**	1.98 **2.60**	1.87 **2.41**	1.80 **2.28**	1.69 **2.09**	1.57 **1.88**	1.42 **1.62**	1.19 **1.28**	1.972 **2.601**
300	3.87 **6.72**	3.03 **4.68**	2.64 **3.85**	2.41 **3.38**	2.25 **3.08**	2.13 **2.86**	2.04 **2.70**	1.97 **2.57**	1.86 **2.38**	1.79 **2.24**	1.68 **2.06**	1.55 **1.85**	1.39 **1.59**	1.15 **1.22**	1.968 **2.592**
400	3.86 **6.70**	3.02 **4.66**	2.63 **3.83**	2.40 **3.37**	2.24 **3.06**	2.12 **2.85**	2.03 **2.69**	1.96 **2.56**	1.85 **2.37**	1.78 **2.23**	1.67 **2.04**	1.54 **1.84**	1.38 **1.57**	1.13 **1.19**	1.966 **2.588**
500	3.86 **6.69**	3.01 **4.65**	2.62 **3.82**	2.39 **3.36**	2.23 **3.05**	2.11 **2.84**	2.03 **2.68**	1.96 **2.55**	1.85 **2.36**	1.77 **2.22**	1.66 **2.03**	1.54 **1.83**	1.38 **1.56**	1.11 **1.16**	1.965 **2.586**
1000	3.85 **6.66**	3.00 **4.63**	2.61 **3.80**	2.38 **3.34**	2.22 **3.04**	2.10 **2.82**	2.02 **2.66**	1.95 **2.53**	1.84 **2.34**	1.76 **2.20**	1.65 **2.01**	1.53 **1.81**	1.36 **1.54**	1.08 **1.11**	1.962 **2.581**
∞	3.84 **6.64**	2.99 **4.60**	2.60 **3.78**	2.37 **3.32**	2.21 **3.02**	2.09 **2.80**	2.01 **2.64**	1.94 **2.51**	1.83 **2.32**	1.75 **2.18**	1.64 **1.99**	1.52 **1.78**	1.35 **1.52**	1.00 **1.00**	1.960 **2.576**

TABLE XIV.—PROBABLE ERRORS OF THE COEFFICENT OF CORRELATION FOR VARIOUS NUMBERS OF OBSERVATIONS OR VARIATES (N) AND FOR VARIOUS VALUES OF r

(Specially Calculated)

Number of Observations	Correlation Coefficient r 0.0	0.1	0.2	0.3	0.4	0.5
25	0.1349	0.1336	0.1295	0.1228	0.1133	0.1012
50	0.0954	0.0944	0.0916	0.0868	0.0801	0.0715
75	0.0779	0.0771	0.0748	0.0709	0.0654	0.0584
100	0.0674	0.0668	0.0648	0.0614	0.0567	0.0506
200	0.0477	0.0472	0.0458	0.0434	0.0401	0.0358
300	0.0389	0.0386	0.0374	0.0354	0.0327	0.0292
400	0.0337	0.0334	0.0324	0.0308	0.0283	0.0253
500	0.0302	0.0299	0.0290	0.0274	0.0253	0.0226
600	0.0275	0.0273	0.0264	0.0251	0.0231	0.0207
700	0.0255	0.0252	0.0245	0.0232	0.0214	0.0191
800	0.0239	0.0236	0.0229	0.0217	0.0200	0.0179
900	0.0225	0.0221	0.0216	0.0205	0.0189	0.0169
1000	0.0213	0.0211	0.0205	0.0194	0.0179	0.0160

Number of Observations	0.6	0.7	0.8	0.9	1.0
25	0.0863	0.0688	0.0486	0.0256	0
50	0.0611	0.0487	0.0343	0.0181	0
75	0.0498	0.0397	0.0280	0.0148	0
100	0.0432	0.0344	0.0243	0.0128	0
200	0.0305	0.0243	0.0172	0.0091	0
300	0.0249	0.0199	0.0140	0.0074	0
400	0.0216	0.0172	0.0121	0.0064	0
500	0.0193	0.0154	0.0109	0.0057	0
600	0.0176	0.0140	0.0099	0.0052	0
700	0.0163	0.0130	0.0092	0.0048	0
800	0.0153	0.0122	0.0086	0.0045	0
900	0.0144	0.0115	0.0081	0.0043	0
1000	0.0137	0.0109	0.0077	0.0041	0

TABLE XV.—THE VALUES OF $1 - r^2$ CORRESPONDING TO VALUES OF r FROM .000 TO .999

r	0	1	2	3	4	5	6	7	8	9
.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.01	.9999	.9999	.9999	.9998	.9998	.9998	.9997	.9997	.9997	.9996
.02	.9996	.9996	.9995	.9995	.9994	.9994	.9993	.9993	.9992	.9992
.03	.9991	.9990	.9990	.9989	.9988	.9988	.9987	.9986	.9986	.9985
.04	.9984	.9983	.9982	.9982	.9981	.9980	.9979	.9978	.9977	.9976
.05	.9975	.9974	.9973	.9972	.9971	.9970	.9969	.9968	.9966	.9965
.06	.9964	.9963	.9962	.9960	.9959	.9958	.9956	.9955	.9954	.9952
.07	.9951	.9950	.9948	.9947	.9945	.9944	.9942	.9941	.9939	.9938
.08	.9936	.9934	.9933	.9931	.9929	.9928	.9926	.9924	.9923	.9921
.09	.9919	.9917	.9915	.9914	.9912	.9910	.9908	.9906	.9904	.9902
.10	.9900	.9898	.9896	.9894	.9892	.9890	.9888	.9886	.9883	.9881
.11	.9879	.9877	.9875	.9872	.9870	.9868	.9865	.9863	.9861	.9858
.12	.9856	.9854	.9851	.9849	.9846	.9844	.9841	.9839	.9836	.9834
.13	.9831	.9828	.9826	.9823	.9820	.9818	.9815	.9812	.9810	.9807
.14	.9804	.9801	.9798	.9796	.9793	.9790	.9787	.9784	.9781	.9778
.15	.9775	.9772	.9769	.9766	.9763	.9760	.9757	.9754	.9750	.9747
.16	.9744	.9741	.9738	.9734	.9731	.9728	.9724	.9721	.9718	.9714
.17	.9711	.9708	.9704	.9701	.9697	.9694	.9690	.9687	.9683	.9680
.18	.9676	.9672	.9669	.9665	.9661	.9658	.9654	.9650	.9647	.9643
.19	.9639	.9635	.9631	.9628	.9624	.9620	.9616	.9612	.9608	.9604
.20	.9600	.9596	.9592	.9588	.9584	.9580	.9576	.9572	.9567	.9563
.21	.9559	.9555	.9551	.9546	.9542	.9538	.9533	.9529	.9525	.9520
.22	.9516	.9512	.9507	.9503	.9498	.9494	.9489	.9485	.9480	.9476
.23	.9471	.9466	.9462	.9457	.9452	.9448	.9443	.9438	.9434	.9429
.24	.9424	.9419	.9414	.9410	.9405	.9400	.9395	.9390	.9385	.9380
.25	.9375	.9370	.9365	.9360	.9355	.9350	.9345	.9340	.9334	.9329
.26	.9324	.9319	.9314	.9308	.9303	.9298	.9292	.9287	.9282	.9276
.27	.9271	.9266	.9260	.9255	.9249	.9244	.9238	.9233	.9227	.9222
.28	.9216	.9210	.9205	.9199	.9193	.9188	.9182	.9176	.9171	.9165
.29	.9159	.9153	.9147	.9142	.9136	.9130	.9124	.9118	.9112	.9106
.30	.9100	.9094	.9088	.9082	.9076	.9070	.9064	.9058	.9051	.9045
.31	.9039	.9033	.9027	.9020	.9014	.9008	.9001	.8995	.8989	.8982
.32	.8976	.8970	.8963	.8957	.8950	.8944	.8937	.8931	.8924	.8918
.33	.8911	.8904	.8898	.8891	.8884	.8878	.8871	.8864	.8858	.8851
.34	.8844	.8837	.8830	.8824	.8817	.8810	.8803	.8796	.8789	.8782
.35	.8775	.8768	.8761	.8754	.8747	.8740	.8733	.8726	.8718	.8711
.36	.8704	.8697	.8690	.8682	.8675	.8668	.8660	.8653	.8646	.8638
.37	.8631	.8624	.8616	.8609	.8601	.8594	.8586	.8579	.8571	.8564
.38	.8556	.8548	.8541	.8533	.8525	.8518	.8510	.8502	.8495	.8487
.39	.8479	.8471	.8463	.8456	.8448	.8440	.8432	.8424	.8416	.8408
.40	.8400	.8392	.8384	.8376	.8368	.8360	.8352	.8344	.8335	.8327
.41	.8319	.8311	.8303	.8294	.8286	.8278	.8269	.8261	.8253	.8244
.42	.8236	.8228	.8219	.8211	.8202	.8194	.8185	.8177	.8168	.8160
.43	.8151	.8142	.8134	.8125	.8116	.8108	.8099	.8090	.8082	.8073
.44	.8064	.8055	.8046	.8038	.8029	.8020	.8011	.8002	.7993	.7984
.45	.7975	.7966	.7957	.7948	.7939	.7930	.7921	.7912	.7902	.7893
.46	.7884	.7875	.7866	.7856	.7847	.7838	.7828	.7819	.7810	.7800
.47	.7791	.7782	.7772	.7763	.7753	.7744	.7734	.7725	.7715	.7706
.48	.7696	.7686	.7677	.7667	.7657	.7648	.7638	.7628	.7619	.7609
.49	.7599	.7589	.7579	.7570	.7560	.7550	.7540	.7530	.7520	.7510

TABLE XV.—*Continued*

r	0	1	2	3	4	5	6	7	8	9
.50	.7500	.7490	.7480	.7470	.7460	.7450	.7440	.7430	.7419	.7409
.51	.7399	.7389	.7379	.7368	.7358	.7348	.7337	.7327	.7317	.7306
.52	.7296	.7286	.7275	.7265	.7254	.7244	.7233	.7223	.7212	.7202
.53	.7191	.7180	.7170	.7159	.7148	.7138	.7127	.7116	.7106	.7095
.54	.7084	.7073	.7062	.7052	.7041	.7030	.7019	.7008	.6997	.6986
.55	.6975	.6964	.6953	.6942	.6931	.6920	.6909	.6898	.6886	.6875
.56	.6864	.6853	.6842	.6830	.6819	.6808	.6796	.6785	.6774	.6762
.57	.6751	.6740	.6728	.6717	.6705	.6694	.6682	.6671	.6659	.6648
.58	.6636	.6624	.6613	.6601	.6589	.6578	.6566	.6554	.6543	.6531
.59	.6519	.6507	.6495	.6484	.6472	.6460	.6448	.6436	.6424	.6412
.60	.6400	.6388	.6376	.6364	.6352	.6340	.6328	.6316	.6303	.6291
.61	.6279	.6267	.6255	.6242	.6230	.6218	.6205	.6193	.6181	.6168
.62	.6156	.6144	.6131	.6119	.6106	.6094	.6081	.6069	.6056	.6044
.63	.6031	.6018	.6006	.5993	.5980	.5968	.5955	.5942	.5930	.5917
.64	.5904	.5891	.5878	.5866	.5853	.5840	.5827	.5814	.5801	.5788
.65	.5775	.5762	.5749	.5736	.5723	.5710	.5697	.5684	.5670	.5657
.66	.5644	.5631	.5618	.5604	.5591	.5578	.5564	.5551	.5538	.5524
.67	.5511	.5498	.5484	.5471	.5457	.5444	.5430	.5417	.5403	.5390
.68	.5376	.5362	.5349	.5335	.5321	.5308	.5294	.5280	.5267	.5253
.69	.5239	.5225	.5211	.5198	.5184	.5170	.5156	.5142	.5128	.5114
.70	.5100	.5086	.5072	.5058	.5044	.5030	.5016	.5002	.4987	.4973
.71	.4959	.4945	.4931	.4916	.4902	.4888	.4873	.4859	.4845	.4830
.72	.4816	.4802	.4787	.4773	.4758	.4744	.4729	.4715	.4700	.4686
.73	.4671	.4656	.4642	.4627	.4612	.4598	.4583	.4568	.4554	.4539
.74	.4524	.4509	.4494	.4480	.4465	.4450	.4435	.4420	.4405	.4390
.75	.4375	.4360	.4345	.4330	.4315	.4300	.4285	.4270	.4254	.4239
.76	.4224	.4209	.4194	.4178	.4163	.4148	.4132	.4117	.4102	.4086
.77	.4071	.4056	.4040	.4025	.4009	.3994	.3978	.3963	.3947	.3932
.78	.3916	.3900	.3885	.3869	.3853	.3838	.3822	.3806	.3791	.3775
.79	.3759	.3743	.3727	.3712	.3696	.3680	.3664	.3648	.3632	.3616
.80	.3600	.3584	.3568	.3552	.3536	.3520	.3504	.3488	.3471	.3455
.81	.3439	.3423	.3407	.3390	.3374	.3358	.3341	.3325	.3309	.3292
.82	.3276	.3260	.3243	.3227	.3210	.3194	.3177	.3161	.3144	.3128
.83	.3111	.3094	.3078	.3061	.3044	.3028	.3011	.2994	.2978	.2961
.84	.2944	.2927	.2910	.2894	.2877	.2860	.2843	.2826	.2809	.2792
.85	.2775	.2758	.2741	.2724	.2707	.2690	.2673	.2656	.2638	.2621
.86	.2604	.2587	.2570	.2552	.2535	.2518	.2500	.2483	.2466	.2448
.87	.2431	.2414	.2396	.2379	.2361	.2344	.2326	.2309	.2291	.2274
.88	.2256	.2238	.2221	.2203	.2185	.2168	.2150	.2132	.2115	.2097
.89	.2079	.2061	.2043	.2026	.2008	.1990	.1972	.1954	.1936	.1918
.90	.1900	.1882	.1864	.1846	.1828	.1810	.1792	.1774	.1755	.1737
.91	.1719	.1701	.1683	.1664	.1646	.1628	.1609	.1591	.1573	.1554
.92	.1536	.1518	.1499	.1481	.1462	.1444	.1425	.1407	.1388	.1370
.93	.1351	.1332	.1314	.1295	.1276	.1258	.1239	.1220	.1202	.1183
.94	.1164	.1145	.1126	.1108	.1089	.1070	.1051	.1032	.1013	.0994
.95	.0975	.0956	.0937	.0918	.0899	.0880	.0861	.0842	.0822	.0803
.96	.0784	.0765	.0746	.0726	.0707	.0688	.0668	.0649	.0630	.0610
.97	.0591	.0572	.0552	.0533	.0513	.0494	.0474	.0455	.0435	.0416
.98	.0396	.0376	.0357	.0337	.0317	.0298	.0278	.0258	.0239	.0219
.99	.0199	.0179	.0159	.0140	.0120	.0100	.0080	.0060	.0040	.0020

TABLE XVI.—VALUES OF $\sqrt{1 - r^2}$ CORRESPONDING TO VALUES OF r FROM .000 TO .999

r	0	1	2	3	4	5	6	7	8	9
.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.01	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9998	.9998
.02	.9998	.9998	.9998	.9997	.9997	.9997	.9997	.9996	.9996	.9996
.03	.9995	.9995	.9995	.9995	.9994	.9994	.9994	.9993	.9993	.9992
.04	.9992	.9992	.9991	.9991	.9990	.9990	.9989	.9989	.9988	.9988
.05	.9987	.9987	.9986	.9986	.9985	.9985	.9984	.9984	.9983	.9983
.06	.9982	.9981	.9981	.9980	.9979	.9979	.9978	.9978	.9977	.9976
.07	.9975	.9975	.9974	.9973	.9973	.9972	.9971	.9970	.9970	.9969
.08	.9968	.9967	.9966	.9965	.9965	.9964	.9963	.9962	.9961	.9960
.09	.9959	.9959	.9958	.9957	.9956	.9955	.9954	.9953	.9952	.9951
.10	.9950	.9949	.9948	.9947	.9946	.9945	.9944	.9943	.9942	.9940
.11	.9939	.9938	.9937	.9936	.9935	.9934	.9932	.9931	.9930	.9929
.12	.9928	.9927	.9925	.9924	.9923	.9922	.9920	.9919	.9918	.9916
.13	.9915	.9914	.9912	.9911	.9910	.9908	.9907	.9906	.9904	.9903
.14	.9902	.9900	.9899	.9897	.9896	.9894	.9893	.9891	.9890	.9888
.15	.9887	.9885	.9884	.9882	.9881	.9879	.9878	.9876	.9874	.9873
.16	.9871	.9870	.9868	.9866	.9865	.9863	.9861	.9860	.9858	.9856
.17	.9854	.9853	.9851	.9849	.9847	.9846	.9844	.9842	.9840	.9838
.18	.9837	.9835	.9833	.9831	.9829	.9827	.9825	.9824	.9822	.9820
.19	.9818	.9816	.9814	.9812	.9810	.9808	.9806	.9804	.9802	.9800
.20	.9798	.9796	.9794	.9792	.9790	.9788	.9786	.9783	.9781	.9779
.21	.9777	.9775	.9773	.9771	.9768	.9766	.9764	.9762	.9759	.9757
.22	.9755	.9753	.9750	.9748	.9746	.9744	.9741	.9739	.9737	.9734
.23	.9732	.9730	.9727	.9725	.9722	.9720	.9718	.9715	.9713	.9710
.24	.9708	.9705	.9703	.9700	.9698	.9695	.9693	.9690	.9688	.9685
.25	.9682	.9680	.9677	.9675	.9672	.9669	.9667	.9664	.9661	.9659
.26	.9656	.9653	.9651	.9648	.9645	.9642	.9640	.9637	.9634	.9631
.27	.9629	.9626	.9623	.9620	.9617	.9614	.9612	.9609	.9606	.9603
.28	.9600	.9597	.9594	.9591	.9588	.9585	.9582	.9579	.9576	.9573
.29	.9570	.9567	.9564	.9561	.9558	.9555	.9552	.9549	.9546	.9543
.30	.9539	.9536	.9533	.9530	.9527	.9524	.9520	.9517	.9514	.9511
.31	.9507	.9504	.9501	.9498	.9494	.9491	.9488	.9484	.9481	.9478
.32	.9474	.9471	.9467	.9464	.9461	.9457	.9454	.9450	.9447	.9443
.33	.9440	.9436	.9433	.9429	.9426	.9422	.9419	.9415	.9411	.9408
.34	.9404	.9401	.9397	.9393	.9390	.9386	.9382	.9379	.9375	.9371
.35	.9367	.9364	.9360	.9356	.9352	.9349	.9345	.9341	.9337	.9333
.36	.9330	.9326	.9322	.9318	.9314	.9310	.9306	.9302	.9298	.9294
.37	.9290	.9286	.9282	.9278	.9274	.9270	.9266	.9262	.9258	.9254
.38	.9250	.9246	.9242	.9237	.9233	.9229	.9225	.9221	.9217	.9212
.39	.9208	.9204	.9200	.9195	.9191	.9187	.9183	.9178	.9174	.9170
.40	.9165	.9161	.9156	.9152	.9148	.9143	.9139	.9134	.9130	.9125
.41	.9121	.9116	.9112	.9107	.9103	.9098	.9094	.9089	.9084	.9080
.42	.9075	.9071	.9066	.9061	.9057	.9052	.9047	.9043	.9038	.9033
.43	.9028	.9024	.9019	.9014	.9009	.9004	.8999	.8995	.8990	.8985
.44	.8980	.8975	.8970	.8965	.8960	.8955	.8950	.8945	.8940	.8935
.45	.8930	.8925	.8920	.8915	.8910	.8905	.8900	.8895	.8890	.8884
.46	.8879	.8874	.8869	.8864	.8858	.8853	.8848	.8843	.8837	.8832
.47	.8827	.8821	.8816	.8811	.8805	.8800	.8794	.8789	.8784	.8778
.48	.8773	.8767	.8762	.8756	.8751	.8745	.8740	.8734	.8728	.8723
.49	.8717	.8712	.8706	.8700	.8695	.8689	.8683	.8678	.8672	.8666

TABLE XVI.—*Continued*

r	0	1	2	3	4	5	6	7	8	9
.50	.8660	.8654	.8649	.8643	.8637	.8631	.8625	.8619	.8614	.8608
.51	.8602	.8596	.8590	.8584	.8578	.8572	.8566	.8560	.8554	.8548
.52	.8542	.8536	.8529	.8523	.8517	.8511	.8505	.8499	.8492	.8486
.53	.8480	.8474	.8467	.8461	.8455	.8449	.8442	.8436	.8429	.8423
.54	.8417	.8410	.8404	.8397	.8391	.8384	.8378	.8371	.8365	.8358
.55	.8352	.8345	.8338	.8332	.8325	.8319	.8312	.8305	.8298	.8292
.56	.8285	.8278	.8271	.8265	.8258	.8251	.8244	.8237	.8230	.8223
.57	.8216	.8210	.8203	.8196	.8189	.8182	.8174	.8167	.8160	.8153
.58	.8146	.8139	.8132	.8125	.8118	.8110	.8103	.8096	.8089	.8081
.59	.8074	.8067	.8059	.8052	.8045	.8037	.8030	.8022	.8015	.8007
.60	.8000	.7992	.7985	.7977	.7970	.7962	.7955	.7947	.7939	.7932
.61	.7924	.7916	.7909	.7901	.7893	.7885	.7877	.7870	.7862	.7854
.62	.7846	.7838	.7830	.7822	.7814	.7806	.7798	.7790	.7782	.7774
.63	.7766	.7758	.7750	.7742	.7733	.7725	.7717	.7709	.7700	.7692
.64	.7684	.7675	.7667	.7659	.7650	.7642	.7633	.7625	.7616	.7608
.65	.7599	.7591	.7582	.7574	.7565	.7556	.7548	.7539	.7530	.7521
.66	.7513	.7504	.7495	.7486	.7477	.7468	.7460	.7451	.7442	.7433
.67	.7424	.7415	.7406	.7396	.7387	.7378	.7369	.7360	.7351	.7341
.68	.7332	.7323	.7314	.7304	.7295	.7285	.7276	.7267	.7257	.7248
.69	.7238	.7229	.7219	.7209	.7200	.7190	.7180	.7171	.7161	.7151
.70	.7141	.7132	.7122	.7112	.7102	.7092	.7082	.7072	.7062	.7052
.71	.7042	.7032	.7022	.7012	.7001	.6991	.6981	.6971	.6960	.6950
.72	.6940	.6929	.6919	.6908	.6898	.6887	.6877	.6866	.6856	.6845
.73	.6834	.6824	.6813	.6802	.6791	.6781	.6770	.6759	.6748	.6737
.74	.6726	.6715	.6704	.6693	.6682	.6671	.6659	.6648	.6637	.6626
.75	.6614	.6603	.6592	.6580	.6569	.6557	.6546	.6534	.6523	.6511
.76	.6499	.6488	.6476	.6464	.6452	.6440	.6428	.6416	.6404	.6392
.77	.6380	.6368	.6356	.6344	.6332	.6320	.6307	.6295	.6283	.6270
.78	.6258	.6245	.6233	.6220	.6208	.6195	.6182	.6170	.6157	.6144
.79	.6131	.6118	.6105	.6092	.6079	.6066	.6053	.6040	.6027	.6013
.80	.6000	.5987	.5973	.5960	.5946	.5933	.5919	.5906	.5982	.5878
.81	.5864	.5850	.5837	.5823	.5809	.5795	.5781	.5766	.5752	.5738
.82	.5724	.5709	.5695	.5680	.5666	.5651	.5637	.5622	.5607	.5592
.83	.5578	.5563	.5548	.5533	.5518	.5502	.5487	.5472	.5457	.5441
.84	.5426	.5410	.5395	.5379	.5363	.5348	.5332	.5316	.5300	.5284
.85	.5268	.5252	.5235	.5219	.5203	.5186	.5170	.5153	.5136	.5120
.86	.5103	.5086	.5069	.5052	.5035	.5018	.5000	.4983	.4966	.4948
.87	.4931	.4913	.4895	.4877	.4859	.4841	.4823	.4805	.4787	.4768
.88	.4750	.4731	.4712	.4694	.4675	.4656	.4637	.4618	.4598	.4579
.89	.4560	.4540	.4520	.4501	.4481	.4461	.4441	.4420	.4400	.4379
.90	.4359	.4338	.4317	.4296	.4275	.4254	.4233	.4211	.4190	.4168
.91	.4146	.4124	.4102	.4080	.4057	.4035	.4012	.3989	.3966	.3943
.92	.3919	.3896	.3872	.3848	.3824	.3800	.3775	.3751	.3726	.3701
.93	.3676	.3650	.3625	.3599	.3573	.3546	.3520	.3493	.3466	.3439
.94	.3412	.3384	.3356	.3328	.3299	.3271	.3242	.3212	.3183	.3153
.95	.3122	.3092	.3061	.3030	.2998	.2966	.2934	.2901	.2868	.2834
.96	.2800	.2765	.2730	.2695	.2659	.2622	.2585	.2548	.2510	.2471
.97	.2431	.2391	.2350	.2308	.2265	.2222	.2178	.2132	.2086	.2039
.98	.1990	.1940	.1889	.1836	.1782	.1726	.1667	.1607	.1545	.1479
.99	.1411	.1339	.1262	.1181	.1094	.0999	.0894	.0774	.0632	.0447

TABLE XVII.—GIVING THE VALUES OF $r = 2 \sin \frac{\pi}{6} \rho$ FOR COMPUTED VALUES OF ρ FROM 0.01 TO 1.00

ρ	r	ρ	r	ρ	r	ρ	r
.01	.010	.26	.271	.51	.528	.76	.775
.02	.021	.27	.282	.52	.538	.77	.785
.03	.031	.28	.292	.53	.548	.78	.794
.04	.042	.29	.303	.54	.558	.79	.804
.05	.052	.30	.313	.55	.568	.80	.813
.06	.063	.31	.323	.56	.578	.81	.823
.07	.073	.32	.334	.57	.588	.82	.833
.08	.084	.33	.344	.58	.598	.83	.842
.09	.094	.34	.354	.59	.608	.84	.852
.10	.105	.35	.364	.60	.618	.85	.861
.11	.115	.36	.375	.61	.628	.86	.870
.12	.126	.37	.385	.62	.638	.87	.880
.13	.136	.38	.395	.63	.648	.88	.889
.14	.146	.39	.406	.64	.658	.89	.899
.15	.157	.40	.416	.65	.668	.90	.908
.16	.167	.41	.426	.66	.677	.91	.917
.17	.178	.42	.436	.67	.687	.92	.927
.18	.188	.43	.447	.68	.697	.93	.936
.19	.199	.44	.457	.69	.707	.94	.945
.20	.209	.45	.467	.70	.717	.95	.954
.21	.219	.46	.477	.71	.727	.96	.964
.22	.230	.47	.487	.72	.736	.97	.973
.23	.240	.48	.497	.73	.746	.98	.982
.24	.251	.49	.508	.74	.756	.99	.991
.25	.261	.50	.518	.75	.765	1.00	1.000

TABLE XVIII.—FUNCTIONS OF r_t FOR USE IN COMPUTING THE APPROXIMATE PROBABLE ERROR OF THE TETRACHORIC COEFFICIENT OF CORRELATION FUNCTION

$$0.6745\sqrt{[1 - r_t^2]\left[1 - \left(\frac{\sin^{-1} r_t}{90^\circ}\right)^2\right]}$$

r_t	Function of r_t	r_t	Function of r_t	r_t	Function of r_t	r_t	Function of r_t
.00	.6745	.25	.6445	.50	.5507	.75	.3756
.01	.6744	.26	.6421	.51	.5455	.76	.3662
.02	.6743	.27	.6396	.52	.5401	.77	.3567
.03	.6741	.28	.6369	.53	.5346	.78	.3470
.04	.6737	.29	.6341	.54	.5289	.79	.3369
.05	.6733	.30	.6312	.55	.5231	.80	.3267
.06	.6728	.31	.6282	.56	.5173	.81	.3161
.07	.6722	.32	.6251	.57	.5112	.82	.3053
.08	.6715	.33	.6219	.58	.5051	.83	.2942
.09	.6706	.34	.6186	.59	.4987	.84	.2827
.10	.6698	.35	.6153	.60	.4922	.85	.2710
.11	.6688	.36	.6118	.61	.4856	.86	.2589
.12	.6677	.37	.6081	.62	.4788	.87	.2463
.13	.6665	.38	.6044	.63	.4719	.88	.2334
.14	.6652	.39	.6006	.64	.4649	.89	.2200
.15	.6638	.40	.5966	.65	.4576	.90	.2062
.16	.6623	.41	.5925	.66	.4502	.91	.1918
.17	.6607	.42	.5884	.67	.4427	.92	.1767
.18	.6590	.43	.5840	.68	.4349	.93	.1610
.19	.6573	.44	.5797	.69	.4270	.94	.1445
.20	.6554	.45	.5751	.70	.4189	.95	.1269
.21	.6534	.46	.5705	.71	.4106	.96	.1083
.22	.6514	.47	.5658	.72	.4021	.97	.0880
.23	.6492	.48	.5608	.73	.3935	.98	.0656
.24	.6469	.49	.5558	.74	.3846	.99	.0395
.25	.6445	.50	.5507	.75	.3756	1.00	.0000

TABLE XIX.—TABLE OF LOG Γ FUNCTIONS OF p (see pp. 55–57)

p	0	1	2	3	4	5	6	7	8	9
1.00	....	9750	9500	9251	9003	8755	8509	8263	8017	7773
1.01	9.997529	7285	7043	6801	6560	6320	6080	5841	5602	5365
1.02	5128	4892	4656	4421	4187	3953	3721	3489	3257	3026
1.03	2796	2567	2338	2110	1883	1656	1430	1205	0981	0757
1.04	0533	0311	0089	$\bar{9}868$	$\bar{9}647$	$\bar{9}427$	$\bar{9}208$	$\bar{8}989$	$\bar{8}772$	$\bar{8}554$
1.05	9.988338	8122	7907	7692	7478	7265	7052	6841	6629	6419
1.06	6209	6000	5791	5583	5376	5169	4963	4758	4553	4349
1.07	4145	3943	3741	3539	3338	3138	2939	2740	2541	2344
1.08	2147	1951	1755	1560	1365	1172	0978	0786	0594	0403
1.09	0212	0022	$\bar{9}833$	$\bar{9}644$	$\bar{9}456$	$\bar{9}269$	$\bar{9}082$	$\bar{8}896$	$\bar{8}710$	$\bar{8}525$
1.10	9.978341	8157	7974	7791	7610	7428	7248	7068	6888	6709
1.11	6531	6354	6177	6000	5825	5650	5475	5301	5128	4955
1.12	4783	4612	4441	4271	4101	3932	3764	3596	3429	3262
1.13	3096	2931	2766	2602	2438	2275	2113	1951	1790	1629
1.14	1469	1309	1150	0992	0835	0677	0521	0365	0210	0055
1.15	9.969901	9747	9594	9442	9290	9139	8988	8838	8688	8539
1.16	8390	8243	8096	7949	7803	7658	7513	7369	7225	7082
1.17	6939	6797	6655	6514	6374	6234	6095	5957	5818	5681
1.18	5544	5408	5272	5137	5002	4868	4734	4601	4469	4337
1.19	4205	4075	3944	3815	3686	3557	3429	3302	3175	3048
1.20	2922	2797	2672	2548	2425	2302	2179	2057	1936	1815
1.21	1695	1575	1456	1337	1219	1101	0984	0867	0751	0636
1.22	0521	0407	0293	0180	0067	9955	8843	9732	9621	9511
1.23	9.959401	9292	9184	9076	8968	8861	8755	8649	8544	8439
1.24	8335	8231	8128	8025	7923	7821	7720	7620	7520	7420
1.25	7321	7223	7125	7027	6930	6834	6738	6642	6547	6453
1.26	6359	6267	6173	6081	5989	5898	5807	5716	5627	5537
1.27	5449	5360	5273	5185	5099	5013	4927	4842	4757	4673
1.28	4589	4506	4423	4341	4259	4178	4097	4017	3938	3858
1.29	3780	3702	3624	3547	3470	3394	3318	3243	3168	3094
1.30	3020	2947	2874	2802	2730	2659	2588	2518	2448	2379
1.31	2310	2242	2174	2106	2040	1973	1907	1842	1777	1712
1.32	1648	1585	1522	1459	1397	1336	1275	1214	1154	1094
1.33	1035	0977	0918	0861	0803	0747	0690	0634	0579	0524
1.34	0470	0416	0362	0309	0257	0205	0153	0102	0051	0001
1.35	9.949951	9902	9853	9805	9757	9710	9663	9617	9571	9525
1.36	9480	9435	9391	9348	9304	9262	9219	9178	9136	9095
1.37	9054	9015	8975	8936	8898	8859	8822	8785	8748	8711
1.38	8676	8640	8605	8571	8537	8503	8470	8437	8405	8373
1.39	8342	8311	8280	8250	8221	8192	8163	8135	8107	8080
1.40	8053	8026	8000	7975	7950	7925	7901	7877	7854	7831
1.41	7808	7786	7765	7744	7723	7703	7683	7664	7645	7626
1.42	7608	7590	7573	7556	7540	7524	7509	7494	7479	7465
1.43	7451	7438	7425	7413	7401	7389	7378	7368	7358	7348
1.44	7338	7329	7321	7312	7305	7298	7291	7284	7278	7273
1.45	7268	7263	7259	7255	7251	7248	7246	7244	7242	7241
1.46	7240	7239	7239	7240	7241	7242	7243	7245	7248	7251
1.47	7254	7258	7262	7266	7271	7277	7282	7289	7295	7302
1.48	7310	7317	7326	7334	7343	7353	7363	7373	7384	7395
1.49	7407	7419	7431	7444	7457	7471	7485	7499	7515	7529

TABLE XIX.—*Continued*

p	0	1	2	3	4	5	6	7	8	9
1.50	9.947545	7561	7577	7594	7612	7629	7647	7666	7685	7704
1.51	7724	7744	7764	7785	7806	7828	7850	7873	7896	7919
1.52	7943	7967	7991	8016	8041	8067	8093	8120	8146	8174
1.53	8201	8229	8258	8287	8316	8346	8376	8406	8437	8468
1.54	8500	8532	8564	8597	8630	8664	8698	8732	8767	8802
1.55	8837	8873	8910	8946	8983	9021	9059	9097	9135	9174
1.56	9214	9254	9294	9334	9375	9417	9458	9500	9543	9586
1.57	9629	9672	9716	9761	9806	9851	9896	9942	9989	$\overline{0}035$
1.58	9.950082	0130	0177	0225	0274	0323	0372	0422	0472	0522
1.59	0573	0624	0676	0728	0780	0833	0886	0939	0993	1047
1.60	1102	1157	1212	1268	1324	1380	1437	1494	1552	1610
1.61	1668	1727	1786	1845	1905	1965	2025	2086	2147	2209
1.62	2271	2333	2396	2459	2522	2586	2650	2715	2780	2845
1.63	2911	2977	3043	3110	3177	3244	3312	3380	3449	3517
1.64	3587	3656	3726	3797	3867	3938	4010	4081	4154	4226
1.65	4299	4372	4446	4519	4594	4668	4743	4819	4894	4970
1.66	5047	5124	5201	5278	5356	5434	5513	5592	5671	5750
1.67	5830	5911	5991	6072	6154	6235	6317	6400	6482	6566
1.68	6649	6733	6817	6901	6986	7072	7157	7243	7329	7416
1.69	7503	7590	7678	7766	7854	7943	8032	8122	8211	8301
1.70	8391	8482	8573	8664	8756	8848	8941	9034	9127	9220
1.71	9314	9409	9502	9598	9693	9788	9884	9980	$\overline{0}077$	$\overline{0}174$
1.72	9.960271	0369	0467	0565	0664	0763	0862	0961	1061	1162
1.73	1262	1363	1464	1566	1668	1770	1873	1976	2079	2183
1.74	2287	2391	2496	2601	2706	2812	2918	3024	3131	3238
1.75	3345	3453	3561	3669	3778	3887	3996	4105	4215	4326
1.76	4436	4547	4659	4770	4882	4994	5107	5220	5333	5447
1.77	5561	5675	5789	5904	6019	6135	6251	6367	6484	6600
1.78	6718	6835	6953	7071	7189	7308	7427	7547	7666	7787
1.79	7907	8028	8149	8270	8392	8514	8636	8759	8882	9005
1.80	9129	9253	9377	9501	9626	9751	9877	$\overline{0}003$	$\overline{0}129$	$\overline{0}255$
1.81	9.970383	0509	0637	0765	0893	1021	1150	1279	1408	1538
1.82	1668	1798	1929	2060	2191	2322	2454	2586	2719	2852
1.83	2985	3118	3252	3386	3520	3655	3790	3925	4061	4197
1.84	4333	4470	4606	4744	4881	5019	5157	5295	5434	5573
1.85	5712	5852	5992	6132	6273	6414	6555	6697	6838	6980
1.86	7123	7266	7408	7552	7696	7840	7984	8128	8273	8419
1.87	8564	8710	8856	9002	9149	9296	9443	9591	9739	9887
1.88	9.980036	0184	0333	0483	0633	0783	0933	1084	1234	1386
1.89	1537	1689	1841	1994	2147	2299	2453	2607	2761	2915
1.90	3069	3224	3379	3535	3690	3846	4003	4159	4316	4474
1.91	4631	4789	4947	5105	5264	5423	5582	5742	5902	6062
1.92	6223	6383	6544	6706	6867	7029	7192	7354	7517	7680
1.93	7844	8007	8171	8336	8500	8665	8830	8996	9161	9327
1.94	9494	9660	9827	9995	$\overline{0}162$	$\overline{0}330$	$\overline{0}498$	$\overline{0}666$	$\overline{0}835$	1004
1.95	9.991173	1343	1512	1683	1853	2024	2195	2366	2537	2709
1.96	2881	3054	3227	3399	3573	3746	3920	4094	4269	4443
1.97	4618	4794	4969	5145	5321	5498	5674	5851	6029	6206
1.98	6384	6562	6740	6919	7098	7277	7457	7637	7817	7997
1.99	8178	8359	8540	8722	8903	9085	9268	9450	9633	9816

TABLE XX.—VALUE OF Q FOR VARIOUS-SIZED FAMILIES AND VARIOUS CROSSOVER VALUES

Number of children	Crossover value									
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$s =$ 2–3	0.0475	0.0900	0.1275	0.1600	0.1875	0.2100	0.2275	0.2400	0.2475	0.2500
$s =$ 4–5	0.0498	0.0981	0.1237	0.1856	0.2226	0.2541	0.2792	0.2976	0.3088	0.3125
$s =$ 6–7	0.0499	0.0996	0.1479	0.1938	0.2358	0.2726	0.3028	0.3252	0.3391	0.3437
$s =$ 8–9	0.0500	0.0999	0.1492	0.1971	0.2420	0.2823	0.3162	0.3418	0.3578	0.3633
$s =$ 10–11	0.0500	0.1000	0.1496	0.1985	0.2453	0.2881	0.3247	0.3530	0.3708	0.3769
$s =$ 12–13	0.0500	0.1000	0.1498	0.1992	0.2471	0.2917	0.3305	0.3610	0.3805	0.3872
$s =$ 14–15	0.0500	0.1000	0.1499	0.1996	0.2482	0.2940	0.3347	0.3671	0.3880	0.3951
$s =$ 16–17	0.0500	0.1000	0.1499	0.1998	0.2488	0.2957	0.3378	0.3718	0.3940	0.4016

TABLE XXI.—STANDARD ERROR OF Q

Number of children	Crossover value									
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$s = 2$	0.1466	0.1921	0.2179	0.2333	0.2420	0.2468	0.2490	0.2498	0.2500	0.2500
$s = 3$	0.1165	0.1480	0.1639	0.1665	0.1654	0.1609	0.1552	0.1497	0.1456	0.1443
$s = 4$	0.1211	0.1448	0.1658	0.1765	0.1802	0.1788	0.1750	0.1731	0.1668	0.1653
$s = 5$	0.0964	0.1284	0.1452	0.1545	0.1548	0.1467	0.1388	0.1305	0.1239	0.1218
$s = 6$	0.0871	0.1210	0.1408	0.1514	0.1548	0.1527	0.1467	0.1411	0.1337	0.1313
$s = 7$	0.0803	0.1118	0.1294	0.1417	0.1397	0.1353	0.1301	0.1172	0.1094	0.1069
$s = 8$	0.0769	0.1068	0.1241	0.1383	0.1392	0.1392	0.1313	0.1225	0.1151	0.1118
$s = 9$	0.0705	0.0995	0.1156	0.1262	0.1295	0.1271	0.1188	0.1080	0.0989	0.0955
$s = 10$	0.0689	0.0937	0.1119	0.1225	0.1273	0.1268	0.1206	0.1112	0.1092	0.0993

TABLE XXII.—GIVING FOR ANY UNIT CHARACTER THE PROPORTION OF RECESSIVE OFFSPRING TO BE EXPECTED: (*R*) IN RANDOM MATINGS OF DOMINANTS WITH DOMINANTS, AND (*S*) IN RANDOM MATINGS OF DOMINANTS WITH RECESSIVES WHEN *B* GIVES THE PROPORTION OF RECESSIVE INDIVIDUALS IN THE GENERAL POPULATION. ABBREVIATED FROM SNYDER, 1934, pp. 9–17

B	*R*	*S*	*B*	*R*	*S*	*B*	*R*	*S*
.001	.0009	.0307	.062	.0398	.1994	.132	.0710	.2665
.002	.0018	.0428	.064	.0408	.2019	.134	.0718	.2680
.003	.0027	.0519	.066	.0418	.2044	.136	.0726	.2694
.004	.0035	.0595	.068	.0428	.2068	.138	.0734	.2709
.005	.0044	.0660	.070	.0438	.2092	.140	.0742	.2723
.006	.0052	.0719	.072	.0448	.2116	.142	.0749	.2737
.007	.0060	.0772	.074	.0457	.2139	.144	.0756	.2751
.008	.0067	.0821	.076	.0467	.2161	.146	.0765	.2765
.009	.0075	.0867	.078	.0477	.2183	.148	.0772	.2778
.010	.0083	.0909	.080	.0486	.2205	.150	.0780	.2792
.012	.0098	.0987	.082	.0496	.2226	.155	.0798	.2825
.014	.0112	.1058	.084	.0505	.2247	.160	.0816	.2857
.016	.0126	.1123	.086	.0514	.2268	.165	.0835	.2889
.018	.0140	.1183	.088	.0524	.2288	.170	.0852	.2919
.020	.0154	.1239	.090	.0533	.2308	.175	.0870	.2950
.022	.0167	.1292	.092	.0542	.2327	.180	.0887	.2979
.024	.0180	.1341	.094	.0551	.2347	.185	.0905	.3008
.026	.0193	.1389	.096	.0560	.2366	.190	.0922	.3036
.028	.0205	.1433	.098	.0568	.2384	.195	.0938	.3063
.030	.0218	.1476	.100	.0577	.2403	.200	.0955	.3090
.032	.0230	.1517	.102	.0586	.2421	.205	.0972	.3117
.034	.0242	.1557	.104	.0595	.2439	.210	.0988	.3143
.036	.0254	.1595	.106	.0603	.2456	.215	.1004	.3168
.038	.0266	.1631	.108	.0612	.2474	.220	.1020	.3193
.040	.0278	.1667	.110	.0621	.2491	.225	.1035	.3217
.042	.0289	.1701	.112	.0629	.2508	.230	.1050	.3241
.044	.0301	.1734	.114	.0637	.2524	.235	.1066	.3265
.046	.0312	.1766	.116	.0646	.2541	.240	.1081	.3288
.048	.0323	.1797	.118	.0654	.2557	.245	.1096	.3311
.050	.0334	.1827	.120	.0662	.2573	.250	.1111	.3333
.052	.0345	.1857	.122	.0670	.2589	.255	.1126	.3355
.054	.0356	.1886	.124	.0678	.2604	.260	.1140	.3377
.056	.0366	.1914	.126	.0686	.2620	.265	.1155	.3398
.058	.0377	.1941	.128	.0694	.2635	.270	.1169	.3419
.060	.0387	.1968	.130	.0702	.2650	.275	.1183	.3440

TABLE XXII.—*Continued*

B	*R*	*S*	*B*	*R*	*S*	*B*	*R*	*S*
.280	.1197	.3460	.455	.1623	.4028	.770	.2185	.4674
.285	.1212	.3481	.460	.1633	.4041	.780	.2200	.4690
.290	.1225	.3500	.465	.1644	.4054	.790	.2215	.4706
.295	.1239	.3520	.470	.1654	.4067	.800	.2229	.4721
.300	.1253	.3539	.475	.1665	.4080	.810	.2244	.4737
.305	.1266	.3558	.480	.1675	.4093	.820	.2258	.4752
.310	.1280	.3577	.485	.1685	.4105	.830	.2273	.4767
.315	.1292	.3595	.490	.1695	.4118	.840	.2287	.4782
.320	.1305	.3613	.500	.1716	.4142	.850	.2301	.4797
.325	.1318	.3631	.510	.1736	.4166	.860	.2316	.4812
.330	.1332	.3649	.520	.1756	.4190	.870	.2329	.4826
.335	.1344	.3666	.530	.1775	.4213	.880	.2343	.4840
.340	.1356	.3683	.540	.1794	.4236	.890	.2356	.4854
.345	.1369	.3700	.550	.1813	.4258	.900	.2370	.4868
.350	.1382	.3717	.560	.1832	.4280	.910	.2383	.4882
.355	.1394	.3734	.570	.1851	.4302	.920	.2397	.4896
.360	.1406	.3750	.580	.1869	.4323	.930	.2410	.4909
.365	.1418	.3766	.590	.1887	.4344	.940	.2424	.4923
.370	.1430	.3782	.600	.1905	.4365	.950	.2436	.4936
.375	.1443	.3798	.610	.1923	.4385	.960	.2449	.4949
.380	.1455	.3814	.620	.1940	.4405	.970	.2462	.4962
.385	.1466	.3829	.630	.1958	.4425	.980	.2475	.4975
.390	.1478	.3844	.640	.1975	.4444	.990	.2487	.4987
.395	.1489	.3859	.650	.1993	.4464	.999	.2499	.4999
.400	.1501	.3874	.660	.2010	.4483			
.405	.1512	.3889	.670	.2026	.4501			
.410	.1524	.3904	.680	.2042	.4519			
.415	.1535	.3918	.690	.2059	.4538			
.420	.1546	.3932	.700	.2075	.4555			
.425	.1557	.3946	.710	.2091	.4573			
.430	.1568	.3960	.720	.2107	.4590			
.435	.1579	.3974	.730	.2123	.4607			
.440	.1590	.3988	.740	.2138	.4624			
.445	.1601	.4002	.750	.2154	.4641			
.450	.1612	.4015	.760	.2170	.4658			

TABLE XXIII.—SQUARES, CUBES, SQUARE ROOTS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
1	1	1	1.0000000	1.0000000	1.000000000
2	4	8	1.4142136	1.2599210	.500000000
3	9	27	1 7320508	1.4422496	.333333333
4	16	64	2.0000000	1.5874011	.250000000
5	25	125	2.2360680	1.7099759	.200000000
6	36	216	2 4494897	1.8171206	.166666667
7	49	343	2.6457513	1.9129312	.142857143
8	64	512	2 8284271	2.0000000	.125000000
9	81	729	3.0000000	2.0800837	.111111111
10	100	1000	3.1622777	2.1544347	.100000000
11	121	1331	3.3166248	2.2239801	.090909091
12	144	1728	3.4641016	2 2894286	.083333333
13	169	2197	3.6055513	2.3513347	.076923077
14	196	2744	3.7416574	2.4101422	.071428571
15	225	3375	3.8729833	2.4662121	.066666667
16	256	4096	4.0000000	2.5198421	.062500000
17	289	4913	4.1231056	2.5712816	.058823529
18	324	5832	4.2426407	2.6207414	.055555556
19	361	6859	4.3588989	2.6684016	.052631579
20	400	8000	4.4721360	2.7144177	.050000000
21	441	9261	4.5825757	2.7589243	.047619048
22	484	10648	4.6904158	2.8020393	.045454545
23	529	12167	4.7958315	2.8438670	.043478261
24	576	13824	4.8989795	2.8844991	.041666667
25	625	15625	5 0000000	2.9240177	.040000000
26	676	17576	5.0990195	2.9624960	.038461538
27	729	19683	5.1961524	3.0000000	.037037037
28	784	21952	5 2915026	3.0365889	.035714286
29	841	24389	5.3851648	3.0723168	.034482759
30	900	27000	5.4772256	3.1072325	.033333333
31	961	29791	5.5677644	3.1413806	.032258065
32	1024	32768	5.6568542	3.1748021	.031250000
33	1089	35937	5 7445626	3.2075343	.030303030
34	1156	39304	5.8309519	3.2396118	.029411765
35	1225	42875	5.9160798	3.2710663	.028571429
36	1296	46656	6.0000000	3.3019272	.027777778
37	1369	50653	6.0827625	3.3322218	.027027027
38	1444	54872	6.1644140	3 3619754	.026315789
39	1521	59319	6.2449980	3.3912114	.025641026
40	1600	64000	6.3245553	3.4199519	.025000000
41	1681	68921	6.4031242	3.4482172	.024390244
42	1764	74088	6 4807407	3.4760266	.023809524
43	1849	79507	6.5574385	3.5033981	.023255814
44	1936	85184	6.6332496	3 5303483	.022727273
45	2025	91125	6.7082039	3 5568933	.022222222
46	2116	97336	6.7823300	3.5830479	.021739130
47	2209	103823	6.8556546	3.6088261	.021276596
48	2304	110592	6 9282032	3.6342411	.020833333
49	2401	117649	7.0000000	3.6593057	.020408163
50	2500	125000	7.0710678	3.6840314	.020000000
51	2601	132651	7.1414284	3.7084298	.019607843
52	2704	140608	7.2111026	3.7325111	.019230769
53	2809	148877	7 2801099	3.7562858	.018867925
54	2916	157464	7.3484692	3 7797631	.018518519
55	3025	166375	7.4161985	3 8029525	.018181818
56	3136	175616	7.4833148	3 8258624	.017857143
57	3249	185193	7 5498344	3.8485011	.017543860
58	3364	195112	7.6157731	3 8708766	.017241379
59	3481	205379	7.6811457	3 8929965	.016949153
60	3600	216000	7 7459667	3 9148676	.016666667
61	3721	226981	7 8102497	3 9364972	.016393443
62	3844	238328	7.8740079	3.9578915	016129032

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
63	3969	250047	7.9372539	3.9790571	.015873016
64	4096	262144	8.0000000	4.0000000	.015625000
65	4225	274625	8.0622577	4.0207256	.015384615
66	4356	287496	8.1240384	4.0412401	.015151515
67	4489	300763	8.1853528	4.0615480	.014925373
68	4624	314432	8.2462113	4.0816551	.014705882
69	4761	328509	8.3066239	4.1015661	.014492754
70	4900	343000	8.3666003	4.1212853	.014285714
71	5041	357911	8.4261498	4.1408178	.014084507
72	5184	373248	8.4852814	4.1601676	.013888889
73	5329	389017	8.5440037	4.1793390	.013698630
74	5476	405224	8.6023253	4.1983364	.013513514
75	5625	421875	8.6602540	4.2171633	.013333333
76	5776	438976	8.7177979	4.2358236	.013157895
77	5929	456533	8.7749644	4.2543210	.012987013
78	6084	474552	8.8317609	4.2726586	.012820513
79	6241	493039	8.8881944	4.2908404	.012658228
80	6400	512000	8.9442719	4.3088695	.012500000
81	6561	531441	9.0000000	4.3267487	.012345679
82	6724	551368	9.0553851	4.3444815	.012195122
83	6889	571787	9.1104336	4.3620707	.012048193
84	7056	592704	9.1651514	4.3795191	.011904762
85	7225	614125	9.2195445	4.3968296	.011764706
86	7396	636056	9.2736185	4.4140049	.011627907
87	7569	658503	9.3273791	4.4310476	.011494253
88	7744	681472	9.3808315	4.4479602	.011363636
89	7921	704969	9.4339811	4.4647451	.011235955
90	8100	729000	9.4868330	4.4814047	.011111111
91	8281	753571	9.5393920	4.4979414	.010989011
92	8464	778688	9.5916630	4.5143574	.010869565
93	8649	804357	9.6436508	4.5306549	.010752688
94	8836	830584	9.6953597	4.5468359	.010638298
95	9025	857375	9.7467943	4.5629026	.010526316
96	9216	884736	9.7979590	4.5788570	.010416667
97	9409	912673	9.8488578	4.5947009	.010309278
98	9604	941192	9.8994949	4.6104363	.010204082
99	9801	970299	9.9498744	4.6260650	.010101010
100	10000	1000000	10.0000000	4.6415888	.010000000
101	10201	1030301	10.0498756	4.6570095	.009900990
102	10404	1061208	10.0995049	4.6723287	.009803922
103	10609	1092727	10.1488916	4.6875482	.009708738
104	10816	1124864	10.1980390	4.7026694	.009615385
105	11025	1157625	10.2469508	4.7176940	.009523810
106	11236	1191016	10.2956301	4.7326235	.009433962
107	11449	1225043	10.3440804	4.7474594	.009345794
108	11664	1259712	10.3923048	4.7622032	.009259259
109	11881	1295029	10.4403065	4.7768562	.009174312
110	12100	1331000	10.4880885	4.7914199	.009090909
111	12321	1367631	10.5356538	4.8058955	.009009009
112	12544	1404928	10.5830052	4.8202845	.008928571
113	12769	1442897	10.6301458	4.8345881	.008849558
114	12996	1481544	10.6770783	4.8488076	.008771930
115	13225	1520875	10.7238053	4.8629442	.008695652
116	13456	1560896	10.7703296	4.8769990	.008620690
117	13689	1601613	10.8166538	4.8909732	.008547009
118	13924	1643032	10.8627805	4.9048681	.008474576
119	14161	1685159	10.9087121	4.9186847	.008403361
120	14400	1728000	10.9544512	4.9324242	.008333333
121	14641	1771561	11.0000000	4.9460874	.008264463
122	14884	1815848	11.0453610	4.9596757	.008196721
123	15129	1860867	11.0905365	4.9731898	.008130081
124	15376	1906624	11.1355287	4.9866310	.008064516

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
125	15625	1953125	11.1803399	5.0000000	.008000000
126	15876	2000376	11.2249722	5.0132979	.007936508
127	16129	2048383	11.2694277	5.0265257	.007874016
128	16384	2097152	11.3137085	5.0396842	.007812500
129	16641	2146689	11.3578167	5.0527743	.007751938
130	16900	2197000	11.4017543	5.0657970	.007692308
131	17161	2248091	11.4455231	5.0787531	.007633588
132	17424	2299968	11.4891253	5.0916434	.007575758
133	17689	2352637	11.5325626	5.1044687	.007518797
134	17956	2406104	11.5758369	5.1172299	.007462687
135	18225	2460375	11.6189500	5.1299278	.007407407
136	18496	2515456	11.6619038	5.1425632	.007352941
137	18769	2571353	11.7046999	5.1551367	.007299270
138	19044	2628072	11.7473401	5.1676493	.007246377
139	19321	2685619	11.7898261	5.1801015	.007194245
140	19600	2744000	11.8321596	5.1924941	.007142857
141	19881	2803221	11.8743421	5.2048279	.007092199
142	20164	2863288	11.9163753	5.2171034	.007042254
143	20449	2924207	11.9582607	5.2293215	.006993007
144	20736	2985984	12.0000000	5.2414828	.006944444
145	21025	3048625	12.0415946	5.2535879	.006896552
146	21316	3112136	12.0830460	5.2656374	.006849315
147	21609	3176523	12.1243557	5.2776321	.006802721
148	21904	3241792	12.1655251	5.2895725	.006756757
149	22201	3307949	12.2065556	5.3014592	.006711409
150	22500	3375000	12.2474487	5.3132928	.006666667
151	22801	3442951	12.2882057	5.3250740	.006622517
152	23104	3511808	12.3288280	5.3368033	.006578947
153	23409	3581577	12.3693169	5.3484812	.006535948
154	23716	3652264	12.4096736	5.3601084	.006493506
155	24025	3723875	12.4498996	5.3716854	.006451613
156	24336	3796416	12.4899960	5.3832126	.006410256
157	24649	3869893	12.5299641	5 3946907	.006369427
158	24964	3944312	12.5698051	5.4061202	.006329114
159	25281	4019679	12.6095202	5.4175015	.006289308
160	25600	4096000	12.6491106	5.4288352	.006250000
161	25921	4173281	12.6885775	5.4401218	.006211180
162	26244	4251528	12.7279221	5.4513618	.006172840
163	26569	4330747	12.7671453	5.4625556	.006134969
164	26896	4410944	12.8062485	5.4737037	.006097561
165	27225	4492125	12.8452326	5.4848066	.006060606
166	27556	4574296	12.8840987	5.4958647	.006024096
167	27889	4657463	12.9228480	5.5068784	.005988024
168	28224	4741632	12.9614814	5.5178484	.005952381
169	28561	4826809	13.0000000	5.5287748	.005917160
170	28900	4913000	13.0384048	5.5396583	.005882353
171	29241	5000211	13.0766968	5 5504991	.005847953
172	29584	5088448	13.1148770	5.5612978	.005813953
173	29929	5177717	13.1529464	5.5720546	.005780347
174	30276	5268024	13.1909060	5.5827702	.005747126
175	30625	5359375	13.2287566	5.5934447	.005714286
176	30976	5451776	13.2664992	5.6040787	.005681818
177	31329	5545233	13.3041347	5.6146724	.005649718
178	31684	5639752	13 3416641	5.6252263	.005617978
179	32041	5735339	13.3790882	5.6357408	.005586592
180	32400	5832000	13.4164079	5.6462162	.005555556
181	32761	5929741	13.4536240	5.6566528	.005524862
182	33124	6028568	13.4907376	5.6670511	.005494505
183	33489	6128487	13.5277493	5.6774114	.005464481
184	33856	6229504	13.5646600	5.6877340	.005434783
185	34225	6331625	13 6014705	5.6980192	.005405405
186	34596	6434856	13.6381817	5.7082675	.005376344

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
187	34969	6539203	13.6747943	5.7184791	.005347594
188	35344	6644672	13.7113092	5.7286543	.005319149
189	35721	6751269	13.7477271	5.7387936	.005291005
190	36100	6859000	13.7840488	5.7488971	.005263158
191	36481	6967871	13.8202750	5.7589652	.005235602
192	36864	7077888	13.8564065	5.7689982	.005208333
193	37249	7189057	13.8924440	5.7789966	.005181347
194	37636	7301384	13.9283883	5.7889604	.005154639
195	38025	7414875	13.9642400	5.7988900	.005128205
196	38416	7529536	14.0000000	5.8087857	.005102041
197	38809	7645373	14.0356688	5.8186479	.005076142
198	39204	7762392	14.0712473	5.8284767	.005050505
199	39601	7880599	14.1067360	5.8382725	.005025126
200	40000	8000000	14.1421356	5.8480355	.005000000
201	40401	8120601	14.1774469	5.8577660	.004975124
202	40804	8242408	14.2126704	5.8674643	.004950495
203	41209	8365427	14.2478068	5.8771307	.004926108
204	41616	8489664	14.2828569	5.8867653	.004901961
205	42025	8615125	14.3178211	5.8963685	.004878049
206	42436	8741816	14.3527001	5.9059406	.004854369
207	42849	8869743	14.3874946	5.9154817	.004830918
208	43264	8998912	14.4222051	5.9249921	.004807692
209	43681	9129329	14.4568323	5.9344721	.004784689
210	44100	9261000	14.4913767	5.9439220	.004761905
211	44521	9393931	14.5258390	5.9533418	.004739336
212	44944	9528128	14.5602198	5.9627320	.004716981
213	45369	9663597	14.5945195	5.9720926	.004694836
214	45796	9800344	14.6287388	5.9814240	.004672897
215	46225	9938375	14.6628783	5.9907264	.004651163
216	46656	10077696	14.6969385	6.0000000	.004629630
217	47089	10218313	14.7309199	6.0092450	.004608295
218	47524	10360232	14.7648231	6.0184617	.004587156
219	47961	10503459	14.7986486	6.0276502	.004566210
220	48400	10648000	14.8323970	6.0368107	.004545455
221	48841	10793861	14.8660687	6.0459435	.004524887
222	49284	10941048	14.8996644	6.0550489	.004504505
223	49729	11089567	14.9331845	6.0641270	.004484305
224	50176	11239424	14.9666295	6.0731779	.004464286
225	50625	11390625	15.0000000	6.0822020	.004444444
226	51076	11543176	15.0332964	6.0911994	.004424779
227	51529	11697083	15.0665192	6.1001702	.004405286
228	51984	11852352	15.0996689	6.1091147	.004385965
229	52441	12008989	15.1327460	6.1180332	.004366812
230	52900	12167000	15.1657509	6.1269257	.004347826
231	53361	12326391	15.1986842	6.1357924	.004329004
232	53824	12487168	15.2315462	6.1446337	.004310345
233	54289	12649337	15.2643375	6.1534495	.004291845
234	54756	12812904	15.2970585	6.1622401	.004273504
235	55225	12977875	15.3297097	6.1710058	.004255319
236	55696	13144256	15.3622915	6.1797466	.004237288
237	56169	13312053	15.3948043	6.1884628	.004219409
238	56644	13481272	15.4272486	6.1971544	.004201681
239	57121	13651919	15.4596248	6.2058218	.004184100
240	57600	13824000	15.4919334	6.2144650	.004166667
241	58081	13997521	15.5241747	6.2230843	.004149378
242	58564	14172488	15.5563492	6.2316797	.004132231
243	59049	14348907	15.5884573	6.2402515	.004115226
244	59536	14526784	15.6204994	6.2487998	.004098361
245	60025	14706125	15.6524758	6.2573248	.004081633
246	60516	14886936	15.6843871	6.2658266	.004065041
247	61009	15069223	15.7162336	6.2743054	.004048583
248	61504	15252992	15.7480157	6.2827613	.004032258

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
249	62001	15438249	15.7797338	6.2911946	.004016064
250	62500	15625000	15.8113883	6.2996053	.004000000
251	63001	15813251	15.8429795	6.3079935	.003984064
252	63504	16003008	15.8745079	6.3163596	.003968254
253	64009	16194277	15.9059737	6.3247035	.003952569
254	64516	16387064	15.9373775	6.3330256	.003937008
255	65025	16581375	15.9687194	6.3413257	.003921569
256	65536	16777216	16.0000000	6.3496042	.003906250
257	66049	16974593	16.0312195	6.3578611	.003891051
258	66564	17173512	16.0623784	6.3660968	.003875969
259	67081	17373979	16.0934769	6.3743111	.003861004
260	67600	17576000	16.1245155	6.3825043	.003846154
261	68121	17779581	16.1554944	6.3906765	.003831418
262	68644	17984728	16.1864141	6.3988279	.003816794
263	69169	18191447	16.2172747	6.4069585	.003802281
264	69696	18399744	16.2480768	6.4150687	.003787879
265	70225	18609625	16.2788206	6.4231583	.003773585
266	70756	18821096	16.3095064	6.4312276	.003759398
267	71289	19034163	16.3401346	6.4392767	.003745318
268	71824	19248832	16.3707055	6.4473057	.003731343
269	72361	19465109	16.4012195	6.4553148	.003717472
270	72900	19683000	16.4316767	6.4633041	.003703704
271	73441	19902511	16.4620776	6.4712736	.003690037
272	73984	20123648	16.4924225	6.4792236	.003676471
273	74529	20346417	16.5227116	6.4871541	.003663004
274	75076	20570824	16.5529454	6.4950653	.003649635
275	75625	20796875	16.5831240	6.5029572	.003636364
276	76176	21024576	16.6132477	6.5108300	.003623188
277	76729	21253933	16.6433170	6.5186839	.003610108
278	77284	21484952	16.6733320	6.5265189	.003597122
279	77841	21717639	16.7032931	6.5343351	.003584229
280	78400	21952000	16.7332005	6.5421326	.003571429
281	78961	22188041	16.7630546	6.5499116	.003558719
282	79524	22425768	16.7928556	6.5576722	.003546099
283	80089	22665187	16.8226038	6.5654144	.003533569
284	80656	22906304	16.8522995	6.5731385	.003521127
285	81225	23149125	16.8819430	6.5808443	.003508772
286	81796	23393656	16.9115345	6.5885323	.003496503
287	82369	23639903	16.9410743	6.5962023	.003484321
288	82944	23887872	16.9705627	6.6038545	.003472222
289	83521	24137569	17.0000000	6.6114890	.003460208
290	84100	24389000	17.0293864	6.6191060	.003448276
291	84681	24642171	17.0587221	6.6267054	.003436426
292	85264	24897088	17.0880075	6.6342874	.003424658
293	85849	25153757	17.1172428	6.6418522	.003412969
294	86436	25412184	17.1464282	6.6493998	.003401361
295	87025	25672375	17.1755640	6.6569302	.003389831
296	87616	25934336	17.2046505	6.6644437	.003378378
297	88209	26198073	17.2336879	6.6719403	.003367003
298	88804	26463592	17.2626765	6.6794200	.003355705
299	89401	26730899	17.2916165	6.6868831	.003344482
300	90000	27000000	17.3205081	6.6943295	.003333333
301	90601	27270901	17.3493516	6.7017593	.003322259
302	91204	27543608	17.3781472	6.7091729	.003311258
303	91809	27818127	17.4068952	6.7165700	.003300330
304	92416	28094464	17.4355958	6.7239508	.003289474
305	93025	28372625	17.4642492	6.7313155	.003278689
306	93636	28652616	17.4928557	6.7386641	.003267974
307	94249	28934443	17.5214155	6.7459967	.003257329
308	94864	29218112	17.5499288	6.7533134	.003246753
309	95481	29503629	17.5783958	6.7606143	.003236246
310	96100	29791000	17.6068169	6.7678995	.003225806

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
311	96721	30080231	17.6351921	6.7751690	.003215434
312	97344	30371328	17.6635217	6.7824229	.003205128
313	97969	30664297	17.6918060	6.7896613	.003194888
314	98596	30959144	17.7200451	6.7968844	.003184713
315	99225	31255875	17.7482393	6.8040921	.003174603
316	99856	31554496	17.7763888	6.8112847	.003164557
317	100489	31855013	17.8044938	6.8184620	.003154574
318	101124	32157432	17.8325545	6.8256242	.003144654
319	101761	32461759	17.8605711	6.8327714	003134796
320	102400	32768000	17.8885438	6.8399037	.003125000
321	103041	33076161	17.9164729	6.8470213	.003115265
322	103684	33386248	17.9443584	6.8541240	.003105590
323	104329	33698267	17.9722008	6.8612120	.003095975
324	104976	34012224	18.0000000	6.8682855	.003086420
325	105625	34328125	18.0277564	6.8753443	.003076923
326	106276	34645976	18.0554701	6.8823888	.003067485
327	106929	34965783	18.0831413	6.8894188	.003058104
328	107584	35287552	18.1107703	6.8964345	.003048780
329	108241	35611289	18.1383571	6.9034359	.003039514
330	108900	35937000	18.1659021	6.9104232	.003030303
331	109561	36264691	18.1934054	6.9173964	.003021148
332	110224	36594368	18.2208672	6.9243556	.003012048
333	110889	36926037	18.2482876	6.9313008	.003003003
334	111556	37259704	18.2756669	6.9382321	.002994012
335	112225	37595375	18.3030052	6.9451496	.002985075
336	112896	37933056	18.3303028	6.9520533	.002976190
337	113569	38272753	18.3575598	6.9589434	.002967359
338	114244	38614472	18.3847763	6.9658198	.002958580
339	114921	38958219	18.4119526	6.9726826	.002949853
340	115600	39304000	18.4390889	6.9795321	.002941176
341	116281	39651821	18.4661853	6.9863681	.002932551
342	116964	40001688	18.4932420	6.9931906	.002923977
343	117649	40353607	18.5202592	7.0000000	.002915452
344	118336	40707584	18.5472370	7.0067962	.002906977
345	119025	41063625	18.5741756	7.0135791	.002898551
346	119716	41421736	18.6010752	7.0203490	.002890173
347	120409	41781923	18.6279360	7.0271058	.002881844
348	121104	42144192	18.6547581	7.0338497	.002873563
349	121801	42508549	18.6815417	7.0405806	.002865330
350	122500	42875000	18.7082869	7.0472987	.002857143
351	123201	43243551	18.7349940	7.0540041	.002849003
352	123904	43614208	18.7616630	7.0606967	.002840909
353	124609	43986977	18.7882942	7.0673767	.002832861
354	125316	44361864	18.8148877	7.0740440	.002824859
355	126025	44738875	18.8414437	7.0806988	.002816901
356	126736	45118016	18.8679623	7.0873411	.002808989
357	127449	45499293	18.8944436	7.0939709	.002801120
358	128164	45882712	18.9208879	7.1005885	.002793296
359	128881	46268279	18.9472953	7.1071937	.002785515
360	129600	46656000	18.9736660	7.1137866	.002777778
361	130321	47045881	19.0000000	7.1203674	.002770083
362	131044	47437928	19.0262976	7.1269360	.002762431
363	131769	47832147	19.0525589	7.1334925	.002754821
364	132496	48228544	19.0787840	7.1400370	.002747253
365	133225	48627125	19.1049732	7.1465695	.002739726
366	133956	49027896	19.1311265	7.1530901	.002732240
367	134689	49430863	19.1572441	7.1595988	.002724796
368	135424	49836032	19.1833261	7.1660957	.002717391
369	136161	50243409	19.2093727	7.1725809	.002710027
370	136900	50653000	19.2353841	7.1790544	.002702703
371	137641	51064811	19.2613603	7.1855162	.002695418
372	138384	51478848	19.2873015	7.1919663	.002688172

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
373	139129	51895117	19.3132079	7.1984050	.002680965
374	139876	52313624	19.3390796	7.2048322	.002673797
375	140625	52734375	19.3649167	7.2112479	.002666667
376	141376	53157376	19.3907194	7.2176522	.002659574
377	142129	53582633	19.4164878	7.2240450	.002652520
378	142884	54010152	19.4422221	7.2304268	.002645503
379	143641	54439939	19.4679223	7.2367972	.002638522
380	144400	54872000	19.4935887	7.2431565	.002631579
381	145161	55306341	19.5192213	7.2495045	.002624672
382	145924	55742968	19.5448203	7.2558415	.002617801
383	146689	56181887	19.5703858	7.2621675	.002610966
384	147456	56623104	19.5959179	7.2684824	.002604167
385	148225	57066625	19.6214169	7.2747864	.002597403
386	148996	57512456	19.6468827	7.2810794	.002590674
387	149769	57960603	19.6723156	7.2873617	.002583979
388	150544	58411072	19.6977156	7.2936330	.002577320
389	151321	58863869	19.7230829	7.2998936	.002570694
390	152100	59319000	19.7484177	7.3061436	.002564103
391	152881	59776471	19.7737199	7.3123828	.002557545
392	153664	60236288	19.7989899	7.3186114	.002551020
393	154449	60698457	19.8242276	7.3248295	.002544529
394	155236	61162984	19.8494332	7.3310369	.002538071
395	156025	61629875	19.8746069	7.3372339	.002531646
396	156816	62099136	19.8997487	7.3434205	.002525253
397	157609	62570773	19.9248588	7.3495966	.002518892
398	158404	63044792	19.9499373	7.3557624	.002512563
399	159201	63521199	19.9749844	7.3619178	.002506266
400	160000	64000000	20.0000000	7.3680630	.002500000
401	160801	64481201	20.0249844	7.3741979	.002493766
402	161604	64964808	20.0499377	7.3803227	.002487562
403	162409	65450827	20.0748599	7.3864373	.002481390
404	163216	65939264	20.0997512	7.3925418	.002475248
405	164025	66430125	20.1246118	7.3986363	.002469136
406	164836	66923416	20.1494417	7.4047206	.002463054
407	165649	67419143	20.1742410	7.4107950	.002457002
408	166464	67917312	20.1990099	7.4168595	.002450980
409	167281	68417929	20.2237484	7.4229142	.002444988
410	168100	68921000	20.2484567	7.4289589	.002439024
411	168921	69426531	20.2731349	7.4349938	.002433090
412	169744	69934528	20.2977831	7.4410189	.002427184
413	170569	70444997	20.3224014	7.4470342	.002421308
414	171396	70957944	20.3469899	7.4530399	.002415459
415	172225	71473375	20.3715488	7.4590359	.002409639
416	173056	71991296	20.3960781	7.4650223	.002403846
417	173889	72511713	20.4205779	7.4709991	.002398082
418	174724	73034632	20.4450483	7.4769664	.002392344
419	175561	73560059	20.4694895	7.4829242	.002386635
420	176400	74088000	20.4939015	7.4888724	.002380952
421	177241	74618461	20.5182845	7.4948113	.002375297
422	178084	75151448	20.5426386	7.5007406	.002369668
423	178929	75686967	20.5669638	7.5066607	.002364066
424	179776	76225024	20 5912603	7.5125715	.002358491
425	180625	76765625	20.6155281	7.5184730	.002352941
426	181476	77308776	20.6397674	7.5243652	.002347418
427	182329	77854483	20.6639783	7.5302482	.002341920
428	183184	78402752	20.6881609	7.5361221	.002336449
429	184041	78953589	20.7123152	7.5419867	.002331002
430	184900	79507000	20.7364414	7.5478423	.002325581
431	185761	80062991	20.7605395	7.5536888	.002320186
432	186624	80621568	20.7846097	7.5595263	.002314815
433	187489	81182737	20.8086520	7.5653548	.002309469
434	188356	81746504	20.8326667	7.5711743	.002304147

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
435	189225	82312875	20.8566536	7.5769849	.002298851
436	190096	82881856	20.8806130	7.5827865	.002293578
437	190969	83453453	20.9045450	7.5885793	.002288330
438	191844	84027672	20.9284495	7.5943633	.002283105
439	192721	84604519	20.9523268	7.6001385	.002277904
440	193600	85184000	20.9761770	7.6059049	.002272727
441	194481	85766121	21.0000000	7.6116626	.002267574
442	195364	86350888	21.0237960	7.6174116	.002262443
443	196249	86938307	21.0475652	7.6231519	.002257336
444	197136	87528384	21.0713075	7.6288837	.002252252
445	198025	88121125	21.0950231	7.6346067	.002247191
446	198916	88716536	21.1187121	7.6403213	.002242152
447	199809	89314623	21.1423745	7.6460272	.002237136
448	200704	89915392	21.1660105	7.6517247	.002232143
449	201601	90518849	21.1896201	7.6574138	.002227171
450	202500	91125000	21.2132034	7.6630943	.002222222
451	203401	91733851	21.2367606	7.6687665	.002217295
452	204304	92345408	21.2602916	7.6744303	.002212389
453	205209	92959677	21.2837967	7.6800857	.002207506
454	206116	93576664	21.3072758	7.6857328	.002202643
455	207025	94196375	21.3307290	7.6913717	.002197802
456	207936	94818816	21.3541565	7.6970023	.002192982
457	208849	95443993	21.3775583	7.7026246	.002188184
458	209764	96071912	21.4009346	7.7082388	.002183406
459	210681	96702579	21.4242853	7.7138448	.002178649
460	211600	97336000	21.4476106	7.7194426	.002173913
461	212521	97972181	21.4709106	7.7250325	.002169197
462	213444	98611128	21.4941853	7.7306141	.002164502
463	214369	99252847	21.5174348	7.7361877	.002159827
464	215296	99897344	21.5406592	7.7417532	.002155172
465	216225	100544625	21.5638587	7.7473109	.002150538
466	217156	101194696	21.5870331	7.7528606	.002145923
467	218089	101847563	21.6101828	7.7584023	.002141328
468	219024	102503232	21.6333077	7.7639361	.002136752
469	219961	103161709	21.6564078	7.7694620	.002132196
470	220900	103823000	21.6794834	7.7749801	.002127660
471	221841	104487111	21.7025344	7.7804904	.002123142
472	222784	105154048	21.7255610	7.7859928	.002118644
473	223729	105823817	21.7485632	7.7914875	.002114165
474	224676	106496424	21.7715411	7.7969745	.002109705
475	225625	107171875	21.7944947	7.8024538	.002105263
476	226576	107850176	21.8174242	7.8079254	.002100840
477	227529	108531333	21.8403297	7.8133892	.002096436
478	228484	109215352	21 8632111	7.8188456	.002092050
479	229441	109902239	21.8860686	7.8242942	.002087683
480	230400	110592000	21.9089023	7.8297353	.002083333
481	231361	111284641	21.9317122	7.8351688	.002079002
482	232324	111980168	21.9544984	7.8405949	.002074689
483	233289	112678587	21.9772610	7.8460134	.002070393
484	234256	113379904	22.0000000	7.8514244	.002066116
485	235225	114084125	22.0227155	7.8568281	.002061856
486	236196	114791256	22.0454077	7.8622242	.002057613
487	237169	115501303	22.0680765	7.8676130	.002053388
488	238144	116214272	22.0907220	7.8729944	.002049180
489	239121	116930169	22.1133444	7.8783684	.002044990
490	240100	117649000	22.1359436	7.8837352	.002040816
491	241081	118370771	22.1585198	7.8890946	.002036660
492	242064	119095488	22.1810730	7.8944468	.002032520
493	243049	119823157	22.2036033	7.8997917	.002028398
494	244036	120553784	22.2261108	7.9051294	.002024291
495	245025	121287375	22.2485955	7.9104599	.002020202
496	246016	122023936	22.2710575	7.9157832	.002016129

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
497	247009	122763473	22.2934968	7.9210994	.002012072
498	248004	123505992	22.3159136	7.9264085	.002008032
499	249001	124251499	22.3383079	7.9317104	.002004008
500	250000	125000000	22.3606798	7.9370053	.002000000
501	251001	125751501	22.3830293	7.9422931	.001996008
502	252004	126506008	22.4053565	7.9475739	.001992032
503	253009	127263527	22.4276615	7.9528477	.001988072
504	254016	128024064	22.4499443	7.9581144	.001984127
505	255025	128787625	22.4722051	7.9633743	.001980198
506	256036	129554216	22.4944438	7.9686271	.001976285
507	257049	130323843	22.5166605	7.9738731	.001972387
508	258064	131096512	22.5388553	7.9791122	.001968504
509	259081	131872229	22.5610283	7.9843444	.001964637
510	260100	132651000	22.5831796	7.9895697	.001960784
511	261121	133432831	22.6053091	7.9947883	.001956947
512	262144	134217728	22.6274170	8.0000000	.001953125
513	263169	135005697	22.6495033	8.0052049	.001949318
514	264196	135796744	22.6715681	8.0104032	.001945525
515	265225	136590875	22.6936114	8.0155946	.001941748
516	266256	137388096	22.7156334	8.0207794	.001937984
517	267289	138188413	22.7376340	8.0259574	.001934236
518	268324	138991832	22.7596134	8.0311287	.001930502
519	269361	139798359	22.7815715	8.0362935	.001926782
520	270400	140608000	22.8035085	8.0414515	.001923077
521	271441	141420761	22.8254244	8.0466030	.001919386
522	272484	142236648	22.8473193	8.0517479	.001915709
523	273529	143055667	22.8691933	8.0568862	.001912046
524	274576	143877824	22.8910463	8.0620180	.001908397
525	275625	144703125	22.9128785	8.0671432	.001904762
526	276676	145531576	22.9346899	8.0722620	.001901141
527	277729	146363183	22.9564806	8.0773743	.001897533
528	278784	147197952	22.9782506	8.0824800	.001893939
529	279841	148035889	23.0000000	8.0875794	.001890359
530	280900	148877000	23.0217289	8.0926723	.001886792
531	281961	149721291	23.0434372	8.0977589	.001883239
532	283024	150568768	23.0651252	8.1028390	.001879699
533	284089	151419437	23.0867928	8.1079128	.001876173
534	285156	152273304	23.1084400	8.1129803	.001872659
535	286225	153130375	23.1300670	8.1180414	.001869159
536	287296	153990656	23.1516738	8.1230962	.001865672
537	288369	154854153	23.1732605	8.1281447	.001862197
538	289444	155720872	23.1948270	8.1331870	.001858736
539	290521	156590819	23.2163735	8.1382230	.001855288
540	291600	157464000	23.2379001	8.1432529	.001851852
541	292681	158340421	23.2594067	8.1482765	.001848429
542	293764	159220088	23.2808935	8.1532939	.001845018
543	294849	160103007	23.3023604	8.1583051	.001841621
544	295936	160989184	23.3238076	8.1633102	.001838235
545	297025	161878625	23.3452351	8.1683092	.001834862
546	298116	162771336	23.3666429	8.1733020	.001831502
547	299209	163667323	23.3880311	8 1782888	.001828154
548	300304	164566592	23.4093998	8.1832695	.001824818
549	301401	165469149	23.4307490	8.1882441	.001821494
550	302500	166375000	23.4520788	8.1932127	.001818182
551	303601	167284151	23.4733892	8.1981752	.001814882
552	304704	168196608	23.4946802	8.2031319	.001811594
553	305809	169112377	23.5159520	8.2080825	.001808318
554	306916	170031464	23.5372046	8.2130271	.001805054
555	308025	170953875	23 5584380	8.2179657	.001801802
556	309136	171879616	23.5796522	8.2228985	.001798561
557	310249	172808693	23.6008474	8.2278254	.001795332
558	311364	173741112	23.6220236	8.2327463	.001792115

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
559	312481	174676879	23.6431808	8.2376614	.001788909
560	313600	175616000	23.6643191	8.2425706	.001785714
561	314721	176558481	23.6854386	8.2474740	.001782531
562	315844	177504328	23.7065392	8.2523715	.001779359
563	316969	178453547	23.7276210	8.2572633	.001776199
564	318096	179406144	23.7486842	8.2621492	.001773050
565	319225	180362125	23.7697286	8.2670294	.001769912
566	320356	181321496	23.7907545	8.2719039	.001766784
567	321489	182284263	23.8117618	8.2767726	.001763668
568	322624	183250432	23.8327506	8.2816355	.001760563
569	323761	184220009	23.8537209	8.2864928	.001757469
570	324900	185193000	23.8746728	8.2913444	.001754386
571	326041	186169411	23.8956063	8.2961903	.001751313
572	327184	187149248	23.9165215	8.3010304	.001748252
573	328329	188132517	23.9374184	8.3058651	.001745201
574	329476	189119224	23.9582971	8.3106941	.001742160
575	330625	190109375	23.9791576	8.3155175	.001739130
576	331776	191102976	24.0000000	8.3203353	.001736111
577	332929	192100033	24.0208243	8.3251475	.001733102
578	334084	193100552	24.0416306	8.3299542	.001730104
579	335241	194104539	24.0624188	8.3347553	.001727116
580	336400	195112000	24.0831891	8.3395509	.001724138
581	337561	196122941	24.1039416	8.3443410	.001721170
582	338724	197137368	24.1246762	8.3491256	.001718213
583	339889	198155287	24.1453929	8.3539047	.001715266
584	341056	199176704	24.1660919	8.3586784	.001712329
585	342225	200201625	24.1867732	8.3634466	.001709402
586	343396	201230056	24.2074369	8.3682095	.001706485
587	344569	202262003	24.2280829	8.3729668	.001703578
588	345744	203297472	24.2487113	8.3777188	.001700680
589	346921	204336469	24.2693222	8.3824653	.001697793
590	348100	205379000	24.2899156	8.3872065	.001694915
591	349281	206425071	24.3104916	8.3919423	.001692047
592	350464	207474688	24.3310501	8.3966729	.001689189
593	351649	208527857	24.3515913	8.4013981	.001686341
594	352836	209584584	24.3721152	8.4061180	.001683502
595	354025	210644875	24.3926218	8.4108326	.001680672
596	355216	211708736	24.4131112	8.4155419	.001677852
597	356409	212776173	24.4335834	8.4202460	.001675042
598	357604	213847192	24.4540385	8.4249448	.001672241
599	358801	214921799	24.4744765	8.4296383	.001669449
600	360000	216000000	24.4948974	8.4343267	.001666667
601	361201	217081801	24.5153013	8.4390098	.001663894
602	362404	218167208	24.5356883	8.4436877	.001661130
603	363609	219256227	24.5560583	8.4483605	.001658375
604	364816	220348864	24.5764115	8.4530281	.001655629
605	366025	221445125	24.5967478	8.4576906	.001652893
606	367236	222545016	24.6170673	8.4623479	.001650165
607	368449	223648543	24.6373700	8.4670001	.001647446
608	369664	224755712	24.6576560	8.4716471	.001644737
609	370881	225866529	24.6779254	8.4762892	.001642036
610	372100	226981000	24.6981781	8.4809261	.001639344
611	373321	228099131	24.7184142	8.4855579	.001636661
612	374544	229220928	24.7386338	8.4901848	.001633987
613	375769	230346397	24.7588368	8.4948065	.001631321
614	376996	231475544	24.7790234	8.4994233	.001628664
615	378225	232608375	24.7991935	8.5040350	.001626016
616	379456	233744896	24.8193473	8.5086417	.001623377
617	380689	234885113	24.8394847	8.5132435	.001620746
618	381924	236029032	24.8596058	8.5178403	.001618123
619	383161	237176659	24.8797106	8.5224321	.001615509
620	384400	238328000	24.8997992	8.5270189	.001612903

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
621	385641	239483061	24.9198716	8.5316009	.001610306
622	386884	240641848	24.9399278	8.5361780	.001607717
623	388129	241804367	24.9599679	8.5407501	.001605136
624	389376	242970624	24.9799920	8.5453173	.001602564
625	390625	244140625	25.0000000	8.5498797	.001600000
626	391876	245314376	25.0199920	8.5544372	.001597444
627	393129	246491883	25.0399681	8.5589899	.001594896
628	394384	247673152	25.0599282	8.5635377	.001592357
629	395641	248858189	25.0798724	8.5680807	.001589825
630	396900	250047000	25.0998008	8.5726189	.001587302
631	398161	251239591	25.1197134	8.5771523	.001584786
632	399424	252435968	25.1396102	8.5816809	.001582278
633	400689	253636137	25.1594913	8.5862047	.001579779
634	401956	254840104	25.1793566	8.5907238	.001577287
635	403225	256047875	25.1992063	8.5952380	.001574803
636	404496	257259456	25.2190404	8.5997476	.001572327
637	405769	258474853	25.2388589	8.6042525	.001569859
638	407044	259694072	25.2586619	8.6087526	.001567398
639	408321	260917119	25.2784493	8.6132480	.001564945
640	409600	262144000	25.2982213	8.6177388	.001562500
641	410881	263374721	25.3179778	8.6222248	.001560062
642	412164	264609288	25.3377189	8.6267063	.001557632
643	413449	265847707	25.3574447	8.6311830	.001555210
644	414736	267089984	25.3771551	8.6356551	.001552795
645	416025	268336125	25.3968502	8.6401226	.001550388
646	417316	269586136	25.4165301	8.6445855	.001547988
647	418609	270840023	25.4361947	8.6490437	.001545595
648	419904	272097792	25.4558441	8.6534974	.001543210
649	421201	273359449	25.4754784	8.6579465	.001540832
650	422500	274625000	25.4950976	8.6623911	.001538462
651	423801	275894451	25.5147016	8.6668310	.001536098
652	425104	277167808	25.5342907	8.6712665	.001533742
653	426409	278445077	25.5538647	8.6756974	.001531394
654	427716	279726264	25.5734237	8.6801237	.001529052
655	429025	281011375	25.5929678	8.6845456	.001526718
656	430336	282300416	25.6124969	8.6889630	.001524390
657	431649	283593393	25.6320112	8.6933759	.001522070
658	432964	284890312	25.6515107	8.6977843	.001519757
659	434281	286191179	25.6709953	8.7021882	.001517451
660	435600	287496000	25.6904652	8.7065877	.001515152
661	436921	288804781	25.7099203	8.7109827	.001512859
662	438244	290117528	25.7293607	8.7153734	.001510574
663	439569	291434247	25.7487864	8.7197596	.001508296
664	440896	292754944	25.7681975	8.7241414	.001506024
665	442225	294079625	25.7875939	8.7285187	.001503759
666	443556	295408296	25.8069758	8.7328918	.001501502
667	444889	296740963	25.8263431	8.7372604	.001499250
668	446224	298077632	25.8456960	8.7416246	.001497006
669	447561	299418309	25.8650343	8.7459846	.001494768
670	448900	300763000	25.8843582	8.7503401	.001492537
671	450241	302111711	25.9036677	8.7546913	.001490313
672	451584	303464448	25.9229628	8.7590383	.001488095
673	452929	304821217	25.9422435	8.7633809	.001485884
674	454276	306182024	25.9615100	8.7677192	.001483680
675	455625	307546875	25.9807621	8.7720532	.001481481
676	456976	308915776	26.0000000	8.7763830	.001479290
677	458329	310288733	26.0192237	8.7807084	.001477105
678	459684	311665752	26.0384331	8.7850296	.001474926
679	461041	313046839	26.0576284	8.7893466	.001472754
680	462400	314432000	26.0768096	8.7936593	.001470588
681	463761	315821241	26.0959767	8.7979679	.001468429
682	465124	317214568	26.1151297	8.8022721	.001466276

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
683	466489	318611987	26.1342687	8.8065722	.001464129
684	467856	320013504	26.1533937	8.8108681	.001461988
685	469225	321419125	26.1725047	8.8151598	.001459854
686	470596	322828856	26.1916017	8.8194474	.001457726
687	471969	324242703	26.2106848	8.8237307	.001455604
688	473344	325660672	26.2297541	8.8280099	.001453488
689	474721	327082769	26.2488095	8.8322850	.001451379
690	476100	328509000	26.2678511	8.8365559	.001449275
691	477481	329939371	26.2868789	8.8408227	.001447178
692	478864	331373888	26.3058929	8.8450854	.001445087
693	480249	332812557	26.3248932	8.8493440	.001443001
694	481636	334255384	26.3438797	8.8535985	.001440922
695	483025	335702375	26.3628527	8.8578489	.001438849
696	484416	337153536	26.3818119	8.8620952	.001436782
697	485809	338608873	26.4007576	8.8663375	.001434720
698	487204	340068392	26.4196896	8.8705757	.001432665
699	488601	341532099	26.4386081	8.8748099	.001430615
700	490000	343000000	26.4575131	8.8790400	.001428571
701	491401	344472101	26.4764046	8.8832661	.001426534
702	492804	345948408	26.4952826	8.8874882	.001424501
703	494209	347428927	26.5141472	8.8917063	.001422475
704	495616	348913664	26.5329983	8.8959204	.001420455
705	497025	350402625	26.5518361	8.9001304	.001418440
706	498436	351895816	26.5706605	8.9043366	.001416431
707	499849	353393243	26.5894716	8.9085387	.001414427
708	501264	354894912	26.6082694	8.9127369	.001412429
709	502681	356400829	26.6270539	8.9169311	.001410437
710	504100	357911000	26.6458252	8.9211214	.001408451
711	505521	359425431	26.6645833	8.9253078	.001406470
712	506944	360944128	26.6833281	8.9294902	.001404494
713	508369	362467097	26.7020598	8.9336687	.001402525
714	509796	363994344	26.7207784	8.9378433	.001400560
715	511225	365525875	26.7394839	8.9420140	.001398601
716	512656	367061696	26.7581763	8.9461809	.001396648
717	514089	368601813	26.7768557	8.9503438	.001394700
718	515524	370146232	26.7955220	8.9545029	.001392758
719	516961	371694959	26.8141754	8.9586581	.001390821
720	518400	373248000	26.8328157	8.9628095	.001388889
721	519841	374805361	26.8514432	8.9669570	.001386963
722	521284	376367048	26.8700577	8.9711007	.001385042
723	522729	377933067	26.8886593	8.9752406	.001383126
724	524176	379503424	26.9072481	8.9793766	.001381215
725	525625	381078125	26.9258240	8.9835089	.001379310
726	527076	382657176	26.9443872	8.9876373	.001377410
727	528529	384240583	26.9629375	8.9917620	.001375516
728	529984	385828352	26.9814751	8.9958829	.001373626
729	531441	387420489	27.0000000	9.0000000	.001371742
730	532900	389017000	27.0185122	9.0041134	.001369863
731	534361	390617891	27.0370117	9.0082229	.001367989
732	535824	392223168	27.0554985	9.0123288	.001366120
733	537289	393832837	27.0739727	9.0164309	.001364256
734	538756	395446904	27.0924344	9.0205293	.001362398
735	540225	397065375	27.1108834	9.0246239	.001360544
736	541696	398688256	27.1293199	9.0287149	.001358696
737	543169	400315553	27.1477439	9.0328021	.001356852
738	544644	401947272	27.1661554	9.0368857	.001355014
739	546121	403583419	27.1845544	9.0409655	.001353180
740	547600	405224000	27.2029410	9.0450417	.001351351
741	549081	406869021	27.2213152	9.0491142	.001349528
742	550564	408518488	27.2396769	9.0531831	.001347709
743	552049	410172407	27.2580263	9.0572482	.001345895
744	553536	411830784	27.2763634	9.0613098	.001344086

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
745	555025	413493625	27.2946881	9.0653677	.001342282
746	556516	415160936	27.3130006	9.0694220	.001340483
747	558009	416832723	27.3313007	9.0734726	.001338688
748	559504	418508992	27.3495887	9.0775197	.001336898
749	561001	420189749	27.3678644	9.0815631	.001335113
750	562500	421875000	27.3861279	9.0856030	.001333333
751	564001	423564751	27.4043792	9.0896392	.001331558
752	565504	425259008	27.4226184	9.0936719	.001329787
753	567009	426957777	27 4408455	9.0977010	.001328021
754	568516	428661064	27 4590604	9.1017265	.001326260
755	570025	430368875	27.4772633	9.1057485	.001324503
756	571536	432081216	27.4954542	9.1097669	.001322751
757	573049	433798093	27.5136330	9.1137818	.001321004
758	574564	435519512	27.5317998	9.1177931	.001319261
759	576081	437245479	27.5499546	9.1218010	.001317523
760	577600	438976000	27.5680975	9 1258053	.001315789
761	579121	440711081	27.5862284	9.1298061	.001314060
762	580644	442450728	27.6043475	9.1338034	.001312336
763	582169	444194947	27.6224546	9.1377971	.001310616
764	583696	445943744	27.6405499	9.1417874	.001308901
765	585225	447697125	27.6586334	9.1457742	.001307190
766	586756	449455096	27.6767050	9.1497576	.001305483
767	588289	451217663	27.6947648	9.1537375	.001303781
768	589824	452984832	27.7128129	9.1577139	.001302083
769	591361	454756609	27.7308492	9.1616869	.001300390
770	592900	456533000	27.7488739	9.1656565	.001298701
771	594441	458314011	27.7668868	9.1696225	.001297017
772	595984	460099648	27.7848880	9.1735852	001295337
773	597529	461889917	27.8028775	9.1775445	.001293661
774	599076	463684824	27.8208555	9.1815003	.001291990
775	600625	465484375	27.8388218	9.1854527	.001290323
776	602176	467288576	27.8567766	9.1894018	.001288660
777	603729	469097433	27.8747197	9 1933474	.001287001
778	605284	470910952	27.8926514	9.1972897	.001285347
779	606841	472729139	27.9105715	9.2012286	.001283697
780	608400	474552000	27.9284801	9.2051641	.001282051
781	609961	476379541	27.9463772	9 2090962	.001280410
782	611524	478211768	27.9642629	9.2130250	001278772
783	613089	480048687	27.9821372	9.2169505	.001277139
784	614656	481890304	28 0000000	9.2208726	.001275510
785	616225	483736625	28.0178515	9.2247914	.001273885
786	617796	485587656	28.0356915	9 2287068	.001272265
787	619369	487443403	28.0535203	9.2326189	.001270648
788	620944	489303872	28.0713377	9 2365277	.001269036
789	622521	491169069	28.0891438	9.2404333	.001267427
790	624100	493039000	28.1069386	9.2443355	.001265823
791	625681	494913671	28.1247222	9.2482344	.001264223
792	627264	496793088	28.1424946	9.2521300	.001262626
793	628849	498677257	28.1602557	9.2560224	.001261034
794	630436	500566184	28.1780056	9.2599114	.001259446
795	632025	502459875	28.1957444	9.2637973	.001257862
796	633616	504358336	28 2134720	9.2676798	.001256281
797	635209	506261573	28.2311884	9.2715592	.001254705
798	636804	508169592	28 2488938	9 2754352	.001253133
799	638401	510082399	28.2665881	9.2793081	.001251564
800	640000	512000000	28 2842712	9.2831777	.001250000
801	641601	513922401	28.3019434	9.2870440	.001248439
802	643204	515849608	28 3196045	9 2909072	.001246883
803	644809	517781627	28.3372546	9.2947671	.001245330
804	646416	519718464	28 3548938	9.2986239	.001243781
805	648025	521660125	28.3725219	9.3024775	.001242236
806	649636	523606616	28.3901391	9.3063278	.001240695

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
807	651249	525557943	28.4077454	9.3101750	.001239157
808	652864	527514112	28.4253408	9.3140190	.001237624
809	654481	529475129	28.4429253	9.3178599	.001236094
810	656100	531441000	28.4604989	9.3216975	.001234568
811	657721	533411731	28.4780617	9.3255320	.001233046
812	659344	535387328	28.4956137	9.3293634	.001231527
813	660969	537367797	28.5131549	9.3331916	.001230012
814	662596	539353144	28.5306852	9.3370167	.001228501
815	664225	541343375	28.5482048	9.3408386	.001226994
816	665856	543338496	28.5657137	9.3446575	.001225490
817	667489	545338513	28.5832119	9.3484731	.001223990
818	669124	547343432	28.6006993	9.3522857	.001222494
819	670761	549353259	28.6181760	9.3560952	.001221001
820	672400	551368000	28.6356421	9.3599016	.001219512
821	674041	553387661	28.6530976	9.3637049	.001218027
822	675684	555412248	28.6705424	9.3675051	.001216545
823	677329	557441767	28.6879766	9.3713022	.001215067
824	678976	559476224	28.7054002	9.3750963	.001213592
825	680625	561515625	28.7228132	9.3788873	.001212121
826	682276	563559976	28.7402157	9.3826752	.001210654
827	683929	565609283	28.7576077	9.3864600	.001209190
828	685584	567663552	28.7749891	9.3902419	.001207729
829	687241	569722789	28.7923601	9.3940206	.001206273
830	688900	571787000	28.8097206	9.3977964	.001204819
831	690561	573856191	28.8270706	9.4015691	.001203369
832	692224	575930368	28.8444102	9.4053387	.001201923
833	693889	578009537	28.8617394	9.4091054	.001200480
834	695556	580093704	28.8790582	9.4128690	.001199041
835	697225	582182875	28.8963666	9.4166297	.001197605
836	698896	584277056	28.9136646	9.4203873	.001196172
837	700569	586376253	28.9309523	9.4241420	.001194743
838	702244	588480472	28.9482297	9.4278936	.001193317
839	703921	590589719	28.9654967	9.4316423	.001191895
840	705600	592704000	28.9827535	9.4353880	.001190476
841	707281	594823321	29.0000000	9.4391307	.001189061
842	708964	596947688	29.0172363	9.4428704	.001187648
843	710649	599077107	29.0344623	9.4466072	.001186240
844	712336	601211584	29.0516781	9.4503410	.001184834
845	714025	603351125	29.0688837	9.4540719	.001183432
846	715716	605495736	29.0860791	9.4577999	.001182033
847	717409	607645423	29.1032644	9.4615249	.001180638
848	719104	609800192	29.1204396	9.4652470	.001179245
849	720801	611960049	29.1376046	9.4689661	.001177856
850	722500	614125000	29.1547595	9.4726824	.001176471
851	724201	616295051	29.1719043	9.4763957	.001175088
852	725904	618470208	29.1890390	9.4801061	.001173709
853	727609	620650477	29.2061637	9.4838136	.001172333
854	729316	622835864	29.2232784	9.4875182	.001170960
855	731025	625026375	29.2403830	9.4912200	.001169591
856	732736	627222016	29.2574777	9.4949188	.001168224
857	734449	629422793	29.2745623	9.4986147	.001166861
858	736164	631628712	29.2916370	9.5023078	.001165501
859	737881	633839779	29.3087018	9.5059980	.001164144
860	739600	636056000	29.3257566	9.5096854	.001162791
861	741321	638277381	29.3428015	9.5133699	.001161440
862	743044	640503928	29.3598365	9.5170515	.001160093
863	744769	642735647	29.3768616	9.5207303	.001158749
864	746496	644972544	29.3938769	9.5244063	.001157407
865	748225	647214625	29.4108823	9.5280794	.001156069
866	749956	649461896	29.4278779	9.5317497	.001154734
867	751689	651714363	29.4448637	9.5354172	.001153403
868	753424	653972032	29.4618397	9.5390818	.001152074

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
869	755161	656234909	29.4788059	9.5427437	.001150748
870	756900	658503000	29.4957624	9.5464027	.001149425
871	758641	660776311	29.5127091	9.5500589	.001148106
872	760384	663054848	29.5296461	9.5537123	.001146789
873	762129	665338617	29.5465734	9.5573630	.001145475
874	763876	667627624	29.5634910	9.5610108	.001144165
875	765625	669921875	29.5803989	9.5646559	.001142857
876	767376	672221376	29.5972972	9.5682982	.001141553
877	769129	674526133	29.6141858	9.5719377	.001140251
878	770884	676836152	29.6310648	9.5755745	.001138952
879	772641	679151439	29.6479342	9.5792085	.001137656
880	774400	681472000	29.6647939	9.5828397	.001136364
881	776161	683797841	29.6816442	9.5864682	.001135074
882	777924	686128968	29.6984848	9.5900939	.001133787
883	779689	688465387	29.7153159	9.5937169	.001132503
884	781456	690807104	29.7321375	9.5973373	.001131222
885	783225	693154125	29.7489496	9.6009548	.001129944
886	784996	695506456	29.7657521	9.6045696	.001128668
887	786769	697864103	29.7825452	9.6081817	.001127396
888	788544	700227072	29.7993289	9.6117911	.001126126
889	790321	702595369	29.8161030	9.6153977	.001124859
890	792100	704969000	29.8328678	9.6190017	.001123596
891	793881	707347971	29.8496231	9.6226030	.001122334
892	795664	709732288	29.8663690	9.6262016	.001121076
893	797449	712121957	29.8831056	9.6297975	.001119821
894	799236	714516984	29.8998328	9.6333907	.001118568
895	801025	716917375	29.9165506	9.6369812	.001117318
896	802816	719323136	29.9332591	9.6405690	.001116071
897	804609	721734273	29.9499583	9.6441542	.001114827
898	806404	724150792	29.9666481	9.6477367	.001113586
899	808201	726572699	29.9833287	9.6513166	.001112347
900	810000	729000000	30.0000000	9.6548938	.001111111
901	811801	731432701	30.0166620	9.6584684	.001109878
902	813604	733870808	30.0333148	9.6620403	.001108647
903	815409	736314327	30.0499584	9.6656096	.001107420
904	817216	738763264	30.0665928	9.6691762	.001106195
905	819025	741217625	30.0832179	9.6727403	.001104972
906	820836	743677416	30.0998339	9.6763017	.001103753
907	822649	746142643	30.1164407	9.6798604	.001102536
908	824464	748613312	30.1330383	9.6834166	.001101322
909	826281	751089429	30.1496269	9.6869701	.001100110
910	828100	753571000	30.1662063	9.6905211	.001098901
911	829921	756058031	30.1827765	9.6940694	.001097695
912	831744	758550528	30.1993377	9.6976151	.001096491
913	833569	761048497	30.2158899	9.7011583	.001095290
914	835396	763551944	30.2324329	9.7046989	.001094092
915	837225	766060875	30.2489669	9.7082369	.001092896
916	839056	768575296	30.2654919	9.7117723	.001091703
917	840889	771095213	30.2820079	9.7153051	.001090513
918	842724	773620632	30.2985148	9 7188354	.001089325
919	844561	776151559	30.3150128	9.7223631	.001088139
920	846400	778688000	30.3315018	9.7258883	.001086957
921	848241	781229961	30.3479818	9.7294109	.001085776
922	850084	783777448	30 3644529	9.7329309	.001084599
923	851929	786330467	30.3809151	9.7364484	.001083424
924	853776	788889024	30.3973683	9.7399634	.001082251
925	855625	791453125	30.4138127	9.7434758	.001081081
926	857476	794022776	30.4302481	9.7469857	.001079914
927	859329	796597983	30.4466747	9.7504930	.001078749
928	861184	799178752	30.4630924	9.7539979	.001077586
929	863041	801765089	30.4795013	9.7575002	.001076426
930	864900	804357000	30.4959014	9.7610001	.001075269

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
931	866761	806954491	30.5122926	9.7644974	.001074114
932	868624	809557568	30.5286750	9.7679922	.001072961
933	870489	812166237	30.5450487	9.7714845	.001071811
934	872356	814780504	30.5614136	9.7749743	.001070664
935	874225	817400375	30.5777697	9.7784616	.001069519
936	876096	820025856	30.5941171	9.7819466	.001068376
937	877969	822656953	30.6104557	9.7854288	.001067236
938	879844	825293672	30.6267857	9.7889087	.001066098
939	881721	827936019	30.6431069	9.7923861	.001064963
940	883600	830584000	30.6594194	9.7958611	.001063830
941	885481	833237621	30.6757233	9.7993336	.001062699
942	887364	835896888	30.6920185	9.8028036	.001061571
943	889249	838561807	30.7083051	9.8062711	.001060445
944	891136	841232384	30.7245830	9.8097362	.001059322
945	893025	843908625	30.7408523	9.8131989	.001058201
946	894916	846590536	30.7571130	9.8166591	.001057082
947	896809	849278123	30.7733651	9.8201169	.001055966
948	898704	851971392	30.7896086	9.8235723	.001054852
949	900601	854670349	30.8058436	9.8270252	.001053741
950	902500	857375000	30.8220700	9.8304757	.001052632
951	904401	860085351	30.8382879	9.8339238	.001051525
952	906304	862801408	30.8544972	9.8373695	.001050420
953	908209	865523177	30.8706981	9.8408127	.001049318
954	910116	868250664	30.8868904	9.8442536	.001048218
955	912025	870983875	30.9030743	9.8476920	.001047120
956	913936	873722816	30.9192497	9.8511280	.001046025
957	915849	876467493	30.9354166	9.8545617	001044932
958	917764	879217912	30.9515751	9.8579929	.001043841
959	919681	881974079	30.9677251	9.8614218	.001042753
960	921600	884736000	30.9838668	9.8648483	.001041667
961	923521	887503681	31.0000000	9.8682724	.001040583
962	925444	890277128	31.0161248	9.8716941	.001039501
963	927369	893056347	31.0322413	9.8751135	.001038422
964	929296	895841344	31.0483494	9.8785305	.001037344
965	931225	898632125	31.0644491	9.8819451	.001036269
966	933156	901428696	31.0805405	9.8853574	.001035197
967	935089	904231063	31.0966236	9.8887673	.001034126
968	937024	907039232	31.1126984	9.8921749	.001033058
969	938961	909853209	31.1287648	9.8955801	.001031992
970	940900	912673000	31.1448230	9.8989830	001030928
971	942841	915498611	31.1608729	9.9023835	001029866
972	944784	918330048	31.1769145	9.9057817	001028807
973	946729	921167317	31.1929479	9.9091776	.001027749
974	948676	924010424	31.2089731	9.9125712	001026694
975	950625	926859375	31.2249900	9.9159624	.001025641
976	952576	929714176	31.2409987	9.9193513	.001024590
977	954529	932574833	31.2569992	9.9227379	001023541
978	956484	935441352	31.2729915	9.9261222	.001022495
979	958441	938313739	31.2889757	9.9295042	.001021450
980	960400	941192000	31.3049517	9.9328839	.001020408
981	962361	944076141	31.3209195	9.9362613	.001019368
982	964324	946966168	31.3368792	9.9396363	.001018330
983	966289	949862087	31.3528308	9.9430092	.001017294
984	968256	952763904	31.3687743	9.9463797	.001016260
985	970225	955671625	31.3847097	9.9497479	.001015228
986	972196	958585256	31.4006369	9.9531138	.001014199
987	974169	961504803	31.4165561	9.9564775	.001013171
988	976144	964430272	31.4324673	9.9598389	.001012146
989	978121	967361669	31.4483704	9.9631981	.001011122
990	980100	970299000	31.4642654	9.9665549	.001010101
991	982081	973242271	31.4801525	9.9699095	.001009082
992	984064	976191488	31.4960315	9.9732619	.001008065

TABLE XXIII.—*Continued*

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
993	986049	979146657	31.5119025	9.9766120	.001007049
994	988036	982107784	31.5277655	9.9799599	.001006036
995	990025	985074875	31.5436206	9.9833055	.001005025
996	992016	988047936	31.5594677	9.9866488	.001004016
997	994009	991026973	31.5753068	9.9899900	.001003009
998	996004	994011992	31.5911380	9.9933289	.001002004
999	998001	997002999	31.6069613	9.9966656	.001001001
1000	1000000	1000000000	31.6227766	10.0000000	.001000000
1001	1002001	1003003001	31.6385840	10.0033322	.0009990010
1002	1004004	1006012008	31.6543836	10.0066622	.0009980040
1003	1006009	1009027027	31.6701752	10.0099899	.0009970090
1004	1008016	1012048064	31.6859590	10.0133155	.0009960159
1005	1010025	1015075125	31.7017349	10.0166389	.0009950249
1006	1012036	1018108216	31.7175030	10.0199601	.0009940358
1007	1014049	1021147343	31.7332633	10.0232791	.0009930487
1008	1016064	1024192512	31.7490157	10.0265958	.0009920635
1009	1018081	1027243729	31.7647603	10.0299104	.0009910803
1010	1020100	1030301000	31.7804972	10.0332228	.0009900990
1011	1022121	1033364331	31.7962262	10.0365330	.0009891197
1012	1024144	1036433728	31.8119474	10.0398410	.0009881423
1013	1026169	1039509197	31.8276609	10.0431469	.0009871668
1014	1028196	1042590744	31.8433666	10.0464506	.0009861933
1015	1030225	1045678375	31.8590646	10.0497521	.0009852217
1016	1032256	1048772096	31.8747549	10.0530514	.0009842520
1017	1034289	1051871913	31.8904374	10.0563485	.0009832842
1018	1036324	1054977832	31.9061123	10.0596435	.0009823183
1019	1038361	1058089859	31.9217794	10.0629364	.0009813543
1020	1040400	1061208000	31.9374388	10.0662271	.0009803922
1021	1042441	1064332261	31.9530906	10.0695156	.0009794319
1022	1044484	1067462648	31.9687347	10.0728020	.0009784736
1023	1046529	1070599167	31.9843712	10.0760863	.0009775171
1024	1048576	1073741824	32.0000000	10.0793684	.0009765625
1025	1050625	1076890625	32.0156212	10.0826484	.0009756098
1026	1052676	1080045576	32.0312348	10.0859262	.0009746589
1027	1054729	1083206683	32.0468407	10.0892019	.0009737098
1028	1056784	1086373952	32.0624391	10.0924755	.0009727626
1029	1058841	1089547389	32.0780298	10.0957469	.0009718173
1030	1060900	1092727000	32.0936131	10.0990163	.0009708738
1031	1062961	1095912791	32.1091887	10.1022835	.0009699321
1032	1065024	1099104768	32.1247568	10.1055487	.0009689922
1033	1067089	1102302937	32.1403173	10.1088117	.0009680542
1034	1069156	1105507304	32.1558704	10.1120726	.0009671180
1035	1071225	1108717875	32.1714159	10.1153314	.0009661836
1036	1073296	1111934656	32.1869539	10.1185882	.0009652510
1037	1075369	1115157653	32.2024844	10.1218428	.0009643202
1038	1077444	1118386872	32.2180074	10.1250953	.0009633911
1039	1079521	1121622319	32.2335229	10 1283457	.0009624639
1040	1081600	1124864000	32.2490310	10.1315941	.0009615385
1041	1083681	1128111921	32.2645316	10.1348403	.0009606148
1042	1085764	1131366088	32.2800248	10.1380845	.0009596929
1043	1087849	1134626507	32.2955105	10.1413266	.0009587728
1044	1089936	1137893184	32.3109888	10.1445667	.0009578544
1045	1092025	1141166125	32.3264598	10.1478047	.0009569378
1046	1094116	1144445336	32.3419233	10.1510406	.0009560229
1047	1096209	1147730823	32.3573794	10.1542744	.0009551098
1048	1098304	1151022592	32.3728281	10.1575062	.0009541985
1049	1100401	1154320649	32.3882695	10.1607359	.0009532888
1050	1102500	1157625000	32.4037035	10.1639636	.0009523810
1051	1104601	1160935651	32.4191301	10.1671893	.0009514748
1052	1106704	1164252608	32.4345495	10.1704129	.0009505703
1053	1108809	1167575877	32 4499615	10.1736344	.0009496676
1054	1110916	1170905464	32.4653662	10.1768539	.0009487666

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